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## Density and Approximation Properties of Weak Markov Systems

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**Abstract.** We study the density and approximation properties of weak Markov systems defined on a closed interval  $[a, b]$ .

**Keywords:** Weak Markov Systems; Density; Relative Differentiation.

### 1. INTRODUCTION

In [10], based on earlier work of Borwein, Bojanov, et al. [1, 2, 3], we studied the density and approximation properties of Markov systems of continuous functions defined on a closed interval  $[a, b]$ . Here we will extend some of the results of [10] to the weak Markov system setting.

This article is organized as follows: in this section we introduce some of the basic definitions and results from the theory of weak Markov systems. For definitions and results not mentioned here, or for clarification of details, the reader is referred to [10], the books [4, 7, 13, 14, 19], and the survey papers [5, 17]. We also present a new representability theorem for weak Markov systems. In Section 2 we generalize to weak Markov systems the differentiation operator introduced in Section 2 of [10]. This generalization is very straightforward, except for the proof of Proposition 3 (a). The results of Section 3 of [10] are extended in Sections 3 and 4: In Section 3 we develop a theory of Chebychev polynomials for weak Markov systems. Although now these polynomials are not necessarily unique, they still have useful and interesting properties. In particular we remark on Theorem 4(d), which shows the existence of disjoint *intervals of equioscillation*. The density results of [10, Section 4], which are given in terms of the distribution of zeros of the Chebychev polynomials, have their counterpart in Section 4 of the present paper. Although the proofs were motivated by arguments used in [10], the task was complicated by the lack of uniqueness of the Chebychev polynomials and by the possibility that the functions in the system may be linearly dependent on a subset of the interval of definition. Finally, in Section 5 we obtain Jackson type theorems. Here the essential idea is to convolve with the Gauss kernel, apply the results of [10] to the Markov systems thus obtained, and then pass to the limit to recover the original system.

Let  $A$  be a set of real numbers, let  $F(A)$  denote the set of all real-valued functions defined on  $A$ , let  $G_n := \{g_0, \dots, g_n\}$  be a sequence of functions, or *system*, and let  $S(G_n)$  denote the linear span of  $\{g_0, \dots, g_n\}$ . A system of functions  $G_n \subset F(A)$  is called a *Chebychev system* or *T-system* if  $A$  contains at least  $n+1$  points, and all the determinants of the square collocation matrices

$$\bigcup \begin{pmatrix} g_0, \dots, g_n \\ t_0, \dots, t_n \end{pmatrix} := \det(g_j(t_i); 0 \leq i, j \leq n)$$

with  $t_0 < \dots < t_n$  in  $A$ , are positive. If all these determinants are merely nonnegative, and, in addition, the functions in  $G_n$  are linearly independent on  $A$ , then  $G_n$  is called a *weak Chebychev system* or *WT-system*. A system  $G_n$  is called a *Markov system* (*weak Markov system*) if  $G_k = \{g_0, \dots, g_k\}$  is a Chebychev system (weak Chebychev system) for each  $k = 0, 1, \dots, n$ . If  $g_0 = 1$ , we say that  $G_n$  is *normalized*. If  $G := \{g_0, g_1, g_2, \dots\} \subset F(A)$  and  $G_n$  is a (normalized) Markov system (weak Markov system) for all  $n \geq 0$ , we say that  $G$  is a (normalized) infinite Markov system (infinite weak Markov system).

Let  $f(t)$  be a real valued function defined on a set  $A$  of  $n \geq 2$  elements. A sequence  $x_0 < \dots < x_{n-1}$  of elements of  $A$  is called a strong alternation of  $f$  of length  $n$ , if either  $(-1)^i f(x_i)$  is positive for all  $i$ , or  $(-1)^i f(x_i)$  is negative for all  $i$ . It is well known that if  $G_n$  is a weak Chebychev system on  $A$ , then no function in  $S(G_n)$  has a strong alternation of length  $n + 2$  on  $A$  [14, 18, 19]. This property will be used in the proof of Theorem 5 below.

Let  $I(A)$  denote the convex hull of  $A$ . We call  $G_n \subset F(A)$  *representable* if for all  $c \in A$  there is a basis  $U_n$  of  $S(G_n)$ , obtained from  $G_n$  by a triangular transformation (i. e.,  $u_0(x) = g_0(x)$  and  $u_i - g_i \in S(g_{i-1}), 1 \leq i \leq n$ ); a strictly increasing function  $h$  (an “embedding function”) defined on  $A$ , with  $h(c) = c$ ; and a set  $P_n := \{p_1, \dots, p_n\}$  of continuous, increasing functions defined on  $I(h(A))$ , such that for every  $t \in A$  and  $1 \leq k \leq n$ ,

$$(1) \quad u_k(x) = u_0(x) \int_c^{h(x)} \int_c^{t_1} \dots \int_c^{t_{k-1}} dp_k(t_k) \dots dp_1(t_1).$$

In this case we say that  $(h, c, P_n, U_n)$  is a *representation* of  $G_n$ . An  $n$ -dimensional linear space  $S_n$  is called *representable*, if it has a representable basis, and  $(h, c, P_n, U_n)$  will be called a *representation for  $S_n$* , if it is a representation for some basis of  $S_n$ .

The main result of [20] implies that a Markov system on an open interval is representable. However not every Markov system on a closed interval is representable. The representability of weak Markov systems can be characterized in terms of the so-called *Condition E* and *property (M)*:

Let  $S(G_n)$  denote the linear span of  $G_n$ . We say that  $G_n$  satisfies *condition E* if for all  $c \in I(A)$  the following two requirements are satisfied:

- (a) If  $G_n$  is linearly independent on  $[c, \infty) \cap A$  then there exists a basis  $\{u_0, \dots, u_n\}$  for  $S(G_n)$ , obtained by a triangular linear transformation, such that for any sequence of integers  $0 \leq k(0) < \dots < k(m) \leq n$ ,  $\{u_{k(r)}\}_{r=0}^m$  is a weak Markov system on  $A \cap [c, \infty)$ .
- (b) If  $G_n$  is linearly independent on  $(-\infty, c] \cap A$  then there exists a basis  $\{v_0, \dots, v_n\}$  for  $G(Z_n)$ , obtained by a triangular linear transformation, such that for any sequence of integers  $0 \leq k(0) < \dots < k(m) \leq n$ ,  $\{(-1)^{r-k(r)} v_{k(r)}\}_{r=0}^m$  is a weak Markov system on  $(-\infty, c] \cap A$ .

Let  $P_n := \{p_1, \dots, p_n\} \subset F(I)$ , where  $I$  is an interval, let  $h$  be a real-valued function defined on  $A$  such that  $h(A) \subset I$ , and let  $x_0 < \dots < x_n$  be points of  $h(A)$ . We say that  $P_n$  satisfies *property (M)* with respect to  $h$  at  $x_0 < \dots < x_n$  if there is a sequence  $\{t_{i,j} : i = 0, \dots, n; j = 0, \dots, n-i\}$  in  $h(A)$  such that

- (a)  $x_j = t_{0,j} (j = 0, \dots, n)$ ;
- (b)  $t_{i,j} < t_{i+1,j} < t_{i,j+1} (i = 0, n-1; j = 0, \dots, n-i)$ ;
- (c) For  $i = 1, \dots, n$ , and  $j = 0, \dots, n-i$  the function  $p_i(x)$  is not constant at  $t_{i,j}$ .

We say that a function  $f$  is not constant at a point  $c \in (a, b)$  if for every  $\epsilon > 0$  there are points  $x_1, x_2 \in (a, b)$  with  $c - \epsilon < x_1 < c < x_2 < c + \epsilon$ , such that  $f(x_1) \neq f(x_2)$ .

**Theorem A.** [16] *Let  $G_n \subset F(A)$ . Then the following statements are equivalent:*

- (a)  $G_n$  is a normalized weak Markov system that satisfies *Condition (E)*.
- (b)  $G_n$  is representable, and there is a representation  $(h, c, P_n, U_n)$  of  $G_n$  such that  $P_n$  satisfies *property (M)* with respect to  $h$  at some sequence  $x_0 < \dots < x_n$  in  $h(A)$ .
- (c)  $G_n$  is representable, and for every representation  $(h, c, P_n, U_n)$  of  $Z_n, P_n$  satisfies *Property (M)* with respect to  $h$  at some sequence  $x_0 < \dots < x_n$  in  $h(A)$ .

The original statement of Theorem A contained two typographical errors, which we have corrected above.

Since Condition E is usually difficult to verify, we give another condition for representability which is general enough for our purposes. It will shed some light on an additional assumption we will make in Section 4.

**Theorem 1.** *Let  $A$  be a set of real numbers such that  $a := \inf A \in A$  and  $b := \sup A \in A$ , and let  $G_n$  be a normalized weak Markov system on  $A$ . Then  $G_n$  is representable if and only if there are numbers  $\alpha < a$  and  $\beta > b$ , and a weak Markov system  $F_n$  on  $[\alpha, a] \cup A \cup [b, \beta]$  such that for each  $0 \leq k \leq n$ ,  $g_k$  is the restriction to  $A$  of  $f_k$ , and the functions in  $F_n$  are linearly independent on each of the intervals  $[\alpha, a]$  and  $[b, \beta]$ .*

*Proof.* If there are numbers  $\alpha < a$  and  $\beta > b$  and a weak Markov system  $F_n$  on  $[\alpha, a] \cup A \cup [b, \beta]$  such that, for each  $0 \leq k \leq n$ ,  $g_k$  is the restriction to  $A$  of  $f_k$ , and the functions in  $F_n$  are linearly independent on each of the intervals  $[\alpha, a]$  and  $[b, \beta]$ , the assertion follows directly from [6, Proposition 5.1 and Theorem 5.8]. To prove the converse, let  $c \in A$ , and let  $(h, c, P_n, U_n)$  be a representation for  $G_n$ . It suffices to prove the assertion for the system  $U_n$ . Let  $r$  be a strictly increasing function on  $[\alpha, a] \cup A \cup [b, \beta]$  that coincides with  $h$  on  $A$ , and for  $1 \leq k \leq n$  let  $q_k$  be a continuous increasing function on  $[r(\alpha), r(\beta)]$ , strictly increasing on each of the intervals  $[\alpha, a]$  and  $[b, \beta]$ , that coincides with  $p_k$  on  $[r(a), r(b)] = [h(a), h(b)]$ . Let  $v_0 = 1$ ,

$$v_1(x) = \int_c^{r(x)} dq_1(t),$$

and

$$v_k(x) = \int_c^{r(x)} \int_c^{t_1} \cdots \int_c^{t_{k-1}} dq_k(t_k) \cdots dq_2(t_2) dq_1(t_1), \quad 2 \leq k \leq n.$$

It is clear that  $v_k = u_k$  for each  $0 \leq k \leq n$ , and from Theorem A or the Lemma of [15] we readily conclude that  $V_n$  is a normalized weak Markov system. Since the functions  $q_k$  are strictly increasing on each of the intervals  $[\alpha, a]$  and  $[b, \beta]$ , a simple inductive argument involving the number of integrations readily shows that the functions in  $V_n$  are linearly independent on each of the intervals  $[\alpha, a]$  and  $[b, \beta]$ .  $\square$

An infinite weak Markov system  $G$  will be called *finitely representable* if  $G_n$  is representable for each  $n > 0$ . At present, it is not known under what conditions an infinite (weak) Markov system defined on a set  $A$  is representable. In other words, the problem of finding conditions under which for every  $c \in A$  there is a strictly increasing function  $h$  defined on  $A$  with  $h(c) = c$ , an infinite sequence  $P := \{p_1, p_2, \dots\}$  of continuous, increasing functions defined on  $I(h(A))$ , and an infinite sequence of functions  $U := \{u_0, u_1, \dots\}$ , such that  $(h, c, P_n, U_n)$  is a representation of  $G_n$  for each  $n > 0$ , is still open.

## 2. RELATIVE DERIVATIVES FOR WEAK MARKOV SYSTEMS

The following are generalizations of [10, Proposition 1 and Proposition 2] and have exactly the same proof.

**Proposition 1.** *Let  $G_n$  be a representable normalized weak Markov system on a set  $A$ ,  $n > 0$ , and let  $(h, c, P_n, U_n)$  be a representation for  $G_n$ . Then  $u_1$  depends only on  $g_1$  and  $c$ . If, moreover,  $p_1(c) = 0$ , then also  $p_1 \circ h$  depends only on  $g_1$  and  $c$ .*

**Proposition 2.** *Let  $G_n$  be a representable normalized weak Markov system of continuous functions on a closed interval  $[a, b]$  with  $n \geq 1$ , and let  $(h, c, P_n, U_n)$  be a representation*

for the restriction to  $(a, b)$  of the functions in  $G_n$ . Then, for  $x \in [a, b]$ ,

$$u_1(x) = \int_c^x dg_1(t)$$

and, if  $n \geq 2$ ,

$$u_k(x) = \int_c^x \int_c^{h(t_1)} \cdots \int_c^{t_{k-1}} dp_k(t_k) \cdots dp_2(t_2) dg_1(t_1), \quad 2 \leq k \leq n.$$

Let  $G_n$  be a representable normalized weak Markov system of continuous functions on a closed interval  $[a, b]$ . A representation  $(h, c, P_n, U_n)$  of  $G_n$  such that  $h$  is continuous at  $a$ , left-continuous on  $(a, b]$ , and  $p_1(c) = 0$ , will be called *standard*. Repeating *verbatim* the discussion in the second paragraph that follows the proof of [10, Proposition 2], we see that every representable weak Markov system of continuous functions on a closed interval  $[a, b]$  has a standard representation for every  $c \in [a, b]$ .

Let  $I$  denote an interval and  $g$  a continuous real-valued strictly increasing function on  $I$ . If  $f$  is a real-valued function on  $I$  and  $x \in I$  then, provided the limit exists, we define:

$$Df(x) := \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{g(x+h) - g(x)}$$

The operator  $D$  is called the *relative derivative* with respect to  $g$ .

Given a representation  $(h, c, P_n, U_n)$  of a weak Markov system  $G_n$  on  $[a, b]$  with  $n \geq 1$ , we define the operator  $H_n$  on  $S(G_n)$  (the *weak relative derivative* with respect to  $(h, c, P_n, U_n)$ ), exactly as in [10]:

$$H_n u_0 := 0, \quad H_n u_1 := 1.$$

If  $n \geq 2$ ,

$$H_n u_2(x) := \int_c^{h(x)} dp_2(t_2).$$

If  $n \geq 3$ ,

$$H_n u_k(x) := \int_c^{h(x)} \int_c^{t_2} \cdots \int_c^{t_{k-1}} dp_k(t_k) \cdots dp_2(t_2), \quad 3 \leq k \leq n.$$

And for every  $f \in S(G_n)$  by linearity.

We then have:

**Proposition 3.** Let  $G := \{1, g_1, g_2, \dots\}$  be a finitely representable normalized infinite weak Markov system on a closed interval  $[a, b]$ , let  $G_n := \{g_0, \dots, g_n\} \subset G$  with  $n \geq 1$ , let  $c \in [a, b]$ , and let  $H_n$  be the weak relative derivative with respect to some standard representation  $(h, c, P_n, U_n)$  of  $G_n$ . Then

- (a)  $\{H_n u_1, H_n u_2, \dots\}$  is a finitely representable normalized infinite weak Markov system on  $[a, b]$ .
- (b) If the functions  $g_k$  are all continuous on  $[a, b]$ , then

$$u_k(x) = u_0(x) \int_c^x H_n u_k(t) dg_1(t), \quad x \in [a, b], \quad 1 \leq k \leq n,$$

- (c) If, moreover,  $g_1$  is strictly increasing, then the operator  $H_n$  depends neither on  $n$  nor on  $c$ , nor on the representation, but only on  $g_1$ .

*Proof.* Let  $(h, c, P_n, U_n)$  be a representation of  $G_n$ . From Theorem A we know that  $P_n$  satisfies property (M) with respect to  $h$  at some sequence  $x_0 < \cdots < x_n$  in  $h[a, b]$ . Let  $\overline{P}_{n-1} := \{p_2, \dots, p_n\}$  and  $\overline{U}_{n-1} := \{H_n u_1, \dots, H_n u_n\}$ ; then  $(h, c, \overline{P}_{n-1}, \overline{U}_{n-1})$  is a representation of  $\{H_n g_1, \dots, H_n g_n\}$ , and we readily see that  $\overline{P}_{n-1}$  satisfies property (M) with



respect to  $h$  at some sequence  $s_0 < \dots < s_n$  in  $h[a, b]$ . Applying again Theorem A, (a) follows.

Part (b) follows directly from Proposition 2. The proof of part (c) is almost identical to that of the corresponding portion of [10, Proposition 3], and will be omitted.  $\square$

Just as in [10], applying [10, Lemma 1], Proposition 3 (instead of [10, Proposition 3]), and bearing in mind the argument used to prove the latter part of Proposition 3, we obtain the following generalizations of [10, Theorem 1 and Theorem 2]:

**Theorem 2.** *Let  $G := \{1, g_1, g_2, \dots\} \subset C([a, b])$  be a finitely representable normalized infinite weak Markov system on  $[a, b]$ , let  $G_n := \{g_0, \dots, g_n\} \subset G$  with  $n \geq 1$ , and assume that  $g_1$  is strictly increasing. Then there is a unique linear operator  $\tilde{D}$  defined on  $S(G)$  and depending only on  $g_1$ , such that if  $(h, c, P_n, U_n)$  is a standard representation of  $G_n$  with associated operator  $H_n$ , then  $\tilde{D} = H_n$  on  $S(G_n)$ .*

As in [10],  $\tilde{D}$  will be called the *generalized derivative* associated with the system  $G$ .

**Theorem 3.** *Let  $G := \{1, g_1, g_2, \dots\} \subset C([a, b])$  be a finitely representable normalized infinite weak Markov system on  $[a, b]$ , let  $G_n := \{g_0, \dots, g_n\} \subset G$  with  $n \geq 1$ , assume that  $g_1$  is strictly increasing, and let  $(h, c, P_n, U_n)$  be any standard representation of  $G_n$ . Then the generalized derivative  $\tilde{D}$  associated with  $G$  has the following properties:*

- (a) *The functions  $\tilde{D}g_k$  are continuous at  $a$ , left-continuous on  $(a, b]$ , and if  $D$  denotes the relative derivative with respect to  $g_1$ , and  $f \in S(G)$ , then  $\tilde{D}f(x) = Df(x)$  a.e. in  $[a, b]$ .*

(b)

$$u_k(x) = u_0(x) \int_c^x \tilde{D}u_k(t) dg_1(t), \quad x \in [a, b], \quad k \geq 1.$$

- (c)  *$\{\tilde{D}g_1, \tilde{D}g_2, \tilde{D}g_3, \dots\}$  is an normalized infinite weak Markov system on  $[a, b]$ .*

- (d) *For any  $n \geq 1$ , if  $(h, c, P_n, U_n)$  is any standard representation of  $G_n$  then*

$$\tilde{D}u_0 = 0, \quad \tilde{D}u_1 = 1.$$

*If  $n \geq 2$ ,*

$$\tilde{D}u_2(x) = \int_c^{h(x)} dp_2(t_2).$$

*If  $n \geq 3$ ,*

$$\tilde{D}u_k(x) = \int_c^{h(x)} \int_c^{t_2} \dots \int_c^{t_{k-1}} dp_k(t_k) \dots dp_2(t_2), \quad 3 \leq k \leq n.$$

We end this section with the following generalization of [10, Proposition 4]. It has the same proof, except that we need to use Theorem 3(b) instead of [10, Theorem 2(b)].

**Proposition 4.** *Let  $G_n$  be a representable normalized weak Markov system of continuous functions on  $[a, b]$  such that  $g_1$  is strictly increasing, and let  $f \in S(G_n)$ . Then, for every  $x_0, x_1 \in [a, b]$ ,*

$$f(x_1) = f(x_0) + \int_{x_0}^{x_1} \tilde{D}f(t) dg_1(t).$$

### 3. GENERALIZED CHEBYCHEV POLYNOMIALS

For the case of a compact interval, Haar's famous unicity theorem says that an  $n$ -dimensional subspace  $S$  of  $C[a, b]$  has a unique element of best approximation in the norm of the supremum for each  $f \in C([a, b])$ , if and only if  $S$  has a basis that is a Chebychev system on  $[a, b]$  [7, 8] (such a space is called a *Chebychev space*). Moreover, if  $S$  is a Chebychev space,  $f \notin S$  and  $g$  is the best approximation to  $f$  in  $S$ , then the function  $e := f - g$  has an equioscillation of length  $n+1$ , i. e., there are points  $a \leq x_0 < \dots < x_n \leq b$  such that

$$e(x_i) = \epsilon(-1)^i \|e\|, \quad i = 0, \dots, n; \quad \epsilon = 1 \quad \text{or} \quad \epsilon = -1.$$

Jones and Karlovitz [11] characterized those finite-dimensional subspaces  $S$  of  $C[a, b]$  having the property that every function  $f \in C[a, b]$  has at least one element of best approximation  $g$  in  $S$  such that the error function  $f - g$  has an equioscillation of length  $n+1$ . This result was generalized to functions defined in more general sets by Deutsch, N uremberg and Singer [9], and was further extended by Kamal. His result, which we will use in the sequel, is the following:

**Theorem B.** [12, Theorem 2.9] *Let  $Q$  be a locally compact totally ordered space that contains at least  $(n+1)$  points, and let  $N$  be an  $n$ -dimensional subspace of  $C_0(Q)$ . Then  $N$  is a weak Chebychev subspace if and only if for each  $f \in C_0(Q)$  there is  $g \in N$  such that  $\|f - g\| = d(f, N)$  and  $f - g$  equioscillates at  $(n+1)$  points of  $Q$ .*

We can now prove

**Proposition 5.** *Let  $G_n$  be a normalized representable weak Markov system in  $C[a, b]$ . Then there is a function  $T_n \in C[a, b]$  such that*

- (a)  $T_n \in S(G_n)$ .
- (b)  $T_n$  has an equioscillation of length  $n+1$ .
- (c)  $\|T_n\| = 1$  and  $T_n(b) > 0$ .

*Proof.* Let  $T_0 = 1$ . If  $n > 0$ , then by Theorem B there is an element of best approximation  $q_n$  to  $g_n$  from  $S(G_{n-1})$ , such that the error function  $g_n - q_n$  has an equioscillation of length  $n+1$ . Setting  $T_n := \alpha_n(g_n - q_n)$ , where  $\alpha_n$  is chosen so that  $\|T_n\| = 1$  and  $T_n(b) > 0$ , the assertion follows.  $\square$

A function that satisfies the conclusions of Proposition 5 will be called a *generalized Chebychev polynomial associated with  $G_n$*  and denoted by  $T_n$ . Note that if  $G_n$  is not a Chebychev system the functions  $T_n$  may not be unique. If  $G$  is an normalized infinite weak Markov system, we may generate a sequence  $\{T_0, T_1, T_2, \dots\}$  by selecting one such  $T_n$  for each integer  $n$ . Such a sequence will be called a *family of generalized Chebychev polynomials associated with  $G$* .

The following theorem should be compared with [10, Corollary 1], which is the corresponding statement for Markov systems.

**Theorem 4.** *Let  $G$  be a normalized infinite weak Markov system in  $C[a, b]$ , and let  $\{T_0, T_1, T_2, \dots\}$  be a family of generalized Chebychev polynomials associated with  $G$ . Then, for each  $n \geq 0$  we have:*

- (a)  $S(\{T_0, \dots, T_n\}) = S(G_n)$
- (b) *If  $y_0 < \dots < y_n$  is an equioscillation for  $T_n$ , then  $T_n$  is monotonic in each interval  $[y_{j-1}, y_j]$ ;  $j = 1, \dots, n$ .*
- (c)  $T_n$  is constant on  $[a, y_0]$  and on  $[y_n, b]$ .
- (d) *There are points  $d_0, d_0^+, \dots, d_n, d_n^+$  such that  $a = d_0 \leq d_0^+ < \dots < d_n \leq d_n^+ = b$  and  $z_0 < \dots < z_n$  is an equioscillation for  $T_n$ , if and only if  $z_i \in [d_i, d_i^+]$ ;  $i = 0, \dots, n$ .*

- (e) There are points  $c_1, c_1^+, \dots, c_n, c_n^+$  such that  $d_{i-1}^+ < c_i \leq c_i^+ < d_i$  for  $1 \leq i \leq n$ , and

$$T_n^{-1}(\{0\}) = \bigcup_{j=1}^n [c_j, c_j^+]$$

- (f) If, moreover,  $G$  is finitely representable, then  $\tilde{D}T_n$  has weakly constant sign in each interval  $[y_{j-1}, y_j]$ ;  $j = 1, \dots, n$ .

*Proof.*

- (a) Trivial: The functions  $T_0, \dots, T_n$  are linearly independent.
- (b) Let us assume for example that  $T_n(y_{j-1}) = -1 = -T_n(y_j)$  and that  $T_n$  is not increasing. Then there are points  $\xi, \eta$ , with  $y_{j-1} < \xi < \eta < y_j$ , such that  $T_n(\xi) > T_n(\eta)$ . Setting  $\delta := (T_n(\eta) + T_n(\xi))/2$  we see that  $y_0, \dots, y_{j-1}, \xi, \eta, y_j, \dots, y_n$  would be a strong alternation of length  $n+3$  for  $T_n - \delta$ , which is a contradiction.
- (c) If  $d < y_0$  and  $T_n(d) = -T_n(y_0)$ , then  $d, y_0, \dots, y_n$  is a strong alternation of length  $n+2$  for  $T_n$ . Otherwise, if  $|T_n(d)| < 1$ , setting  $\delta = (T_n(d) + T_n(y_0))/2$ , we see that  $T_n - \delta$  would have a strong alternation of length  $n+2$ . Thus  $T_n(d) = T_n(y_0)$ . In similar fashion we see that  $T_n(x) = T_n(y_n)$  in  $[y_n, b]$ .
- (d) Let  $y_0 < \dots < y_n$  be an equioscillation for  $T_n$  in  $[a, b]$ . For each  $j = 0, \dots, n$  let  $I_j := \{x \in (y_{j-1}, y_{j+1}) : T_n(x) = T_n(y_j)\}$ , where  $y_{-1} := a$  and  $y_{n+1} := b$ . Let  $d_j := \inf I_j$ , and  $d_j^+ := \sup I_j$ ; in view of (b) and the continuity of  $T_n$  we see that  $I_j = [d_j, d_j^+]$ , whereas (c) implies that  $d_0 = a$  and  $d_n^+ = b$ ; it is also clear that  $d_{j-1} \leq d_{j-1}^+ < d_j \leq d_j^+$  by construction. Moreover, if  $x \in [a, b]$  is such that  $|T_n(x)| = 1$ , bearing in mind that  $x \in [y_{j-1}, y_j]$  for some  $j$ ,  $0 \leq j \leq n+1$ , we conclude that either  $T_n(x) = T_n(y_{j-1})$  or  $T_n(x) = T_n(y_j)$ , whence either  $x \in I_{j-1}$  or  $x \in I_j$ . Therefore

$$\bigcup_{j=0}^n I_j = T_n^{-1}(\{-1, 1\}).$$

Thus, if  $z_0 < \dots < z_n$  is an equioscillation we deduce that  $\{z_0, \dots, z_n\} \subset \bigcup_{j=0}^n I_j$ . Let us assume that for some  $j$ ,  $0 \leq j \leq n$ ,  $z_j \notin I_j$ , and let  $j_0$  the first index for which  $z_{j_0} \notin I_{j_0}$ ; then  $\{z_{j_0}, \dots, z_n\} \subset \bigcup_{j>j_0}^n I_j$ ; this implies that at least two consecutive  $z_j$ 's must belong to the same interval  $I_j$ ; but this contradicts the assumption that  $z_0 < \dots < z_n$  is an equioscillation.

- (e) Since  $T_n(d_{j-1}^+)T_n(d_j) < 0$  for each  $1 \leq j \leq n$ , there is a point  $x \in (d_{j-1}^+, d_j)$  such that  $T_n(x) = 0$ . Thus  $K_j := \{x \in (d_{j-1}^+, d_j) : T_n(x) = 0\} \neq \emptyset$ . Let  $c_j := \inf K_j$  and  $c_j^+ := \sup K_j$ . By continuity  $T_n(c_j) = 0 = T_n(c_j^+)$ , and (b) implies that  $K_j = [c_j, c_j^+]$ . Moreover, since (d) implies that  $T_n$  is constant and nonzero on  $[d_j, d_j^+]$ , it is clear that if  $T_n(x) = 0$  for some  $x \in [a, b]$ , then  $x \notin \bigcup_{j=0}^n [d_j, d_j^+]$ ; therefore  $x \in (d_{j-1}^+, d_j)$  for some  $j$ , i. e.  $x \in K_j$ .
- (f) Let us assume, for instance, that  $T_n(y_{j-1}) = -1 = -T_n(y_j)$ ; therefore (b) implies that  $T_n$  is increasing on  $[y_{j-1}, y_j]$ . If  $\tilde{D}T_n$  is negative in  $(y_{j-1}, y_j)$  there must be a point  $x_1 \in (y_{j-1}, y_j)$  such that  $\tilde{D}T_n(x_1) < 0$ . Since  $\tilde{D}T_n$  is left-continuous, there must be a point  $x_0 \in (y_{j-1}, x_1)$  such that  $\tilde{D}T_n < 0$  in  $[x_0, x_1]$ . Applying Proposition 4 we thus have

$$T_n(x_1) - T_n(x_0) = \int_{x_0}^{x_1} \tilde{D}T_n(s) ds < 0.$$

Since  $T_n$  is increasing on  $[y_{j-1}, y_j]$ , we have obtained a contradiction.  $\square$

The intervals  $[d_j, d_j^+]$  will be called *equioscillation intervals* of  $T_n$ , the intervals  $[c_j, c_j^+]$  will be called *zero intervals* of  $T_n$ , and the left endpoints  $c_j$  of the zero intervals will be called  $\ell$ -zeros of  $T_n$ .

#### 4. DENSITY OF INFINITE WEAK MARKOV SYSTEMS AND ZEROS OF CHEBYCHEV POLYNOMIALS

In this section we will assume that  $G$  is a finitely representable normalized infinite weak Markov system defined on an interval  $[a, b]$ . Clearly  $g_1$  is increasing on  $[a, b]$ . If  $(h, c, P_n, U_n)$  is a representation of  $G_n$ , then  $dp_1 = dg_1$ ; this implies that all the functions in  $S(G_n)$  must be constant on the same subintervals of  $[a, b]$  where  $g_1$  is constant. To obtain density theorems for  $C[a, b]$  we will therefore assume that  $g_1$  is strictly increasing on  $[a, b]$ . Once such a density theorem is obtained, it is easy to obtain a corresponding density theorem valid in the case where  $g_1$  is not strictly increasing. That theorem would obtain for the subset of functions in  $C[a, b]$  that are constant on those subintervals of  $[a, b]$  where  $g_1$  is constant.

Let  $\{T_n\}_{n \geq 1}$  be a sequence of generalized Chebychev polynomials associated with  $G$ . We define

$$M_n := \max\{|c_i - c_{i-1}| : 1 \leq i \leq n+1\},$$

where  $c_1, \dots, c_n$  are the  $\ell$ -zeros of  $T_n$ ,  $c_0 = a$ , and  $c_{n+1} = b$ .

We will also assume that  $G$  has the following property: *There are points  $a_1, b_1$ ,  $a \leq a_1 < b_1 \leq b$  such that for every  $n \geq 1$ ,  $G_n$  is linearly independent in  $[a, a_1] \cup [b_1, b]$ .* Although Theorem 1 implies that such points exist for each  $n$ , they depend on  $n$ , and they may coalesce with the endpoints; for example, we could have  $\lim_{n \rightarrow \infty} a_1 = a$ . With this additional hypothesis, the restriction of the functions in  $G$  to  $[a, a_1] \cup [b_1, b]$  is a normalized infinite weak Markov system, and we obtain the following generalization of [10, Lemma 2]:

**Lemma 1.** *Let  $G$  satisfy the hypotheses of the previous paragraphs, let  $a < a_1 < b_1 < b$ , let  $f$  be the function defined in  $A := [a, a_1] \cup [b_1, b]$  by*

$$f(x) := \begin{cases} 0 & \text{if } x \in [a, a_1] \\ 1 & \text{if } x \in [b_1, b], \end{cases}$$

*and let  $S_n \in S(G_n)$  denote a function whose restriction to  $A$  is an element of best approximation to  $f$  in  $A$  such that  $f - S_n$  has an equioscillation of length  $n+2$  (such a  $S_n$  exists by Theorem B). Then*

- (a)  $S_n$  is increasing on  $[a_1, b_1]$ .
- (b) Assume that  $\lim_{n \rightarrow \infty} M_n = 0$ . Then there is a constant  $K$  such that

$$\|f - S_n\|_A \leq KM_n/(b_1 - a_1),$$

where  $\|\cdot\|_A$  denotes the norm of the supremum in  $A$ .

*Proof.* The proof of (a) is identical to that of [10, Lemma 2], using Theorem B instead of the alternation theorem for Chebychev systems, Proposition 4 instead of [10, Proposition 4], and Theorem 3(c) instead of [10, Theorem 2(c)] to show that  $\{\tilde{D}g_1, \dots, \tilde{D}g_n\}$  is a weak Markov system.

To establish (b) we repeat the steps used in the proof of [10, Lemma 2(b)]. The only difference is that in step (iii) we use  $\ell$ -zeros instead of zeros, and Theorem 4 instead of [10, Corollary 1].  $\square$

**Theorem 5.** *Let  $G \subset C[a, b]$  be a finitely representable normalized infinite weak Markov system such that  $g_1$  is strictly increasing on  $[a, b]$ . Assume there are points  $a_1, b_1$ ,  $a \leq a_1 < b_1 \leq b$ , such that for every  $n \geq 1$ ,  $G_n$  is linearly independent in  $[a, a_1] \cup [b_1, b]$ . Then the following propositions are equivalent:*

- (a)  $S(G)$  is dense in  $C[a, b]$ , in the norm of the supremum.
- (b)  $\lim_{n \rightarrow \infty} M_n = 0$ .

*Proof.* If  $M_n$  does not converge to zero as  $n \rightarrow \infty$ , then there is a number  $r > 0$  such that for each  $k > 0$  we may choose an integer  $n_k > 0$  such that  $M_{n_k} \geq r$ . For each  $k$ , let  $c_{j_k}, c_{j_k+1}$  be two consecutive  $\ell$ -zeros of  $T_{n_k}$  such that  $c_{j_k+1} - c_{j_k} = M_{n_k} \geq r$ , where  $1 \leq j_k \leq n_k$  depends of  $n_k$ . The sequence  $\{c_{j_k} : k \geq 0\}$  will have a subsequence  $\{\alpha_k : k \geq 1\}$  that converges to a point  $\alpha_0$ .

In summation: If  $M_n$  does not converge to zero as  $n \rightarrow \infty$ , then there is a number  $r > 0$  and sequences  $\{r(k) : k \geq 1\}$  and  $\{c_{j_{r(k)}} : k \geq 1\}$ , such that  $c_{j_{r(k)}+1} - c_{j_{r(k)}} = M_{r(k)} \geq r$ , and  $\lim c_{j_{r(k)}} = \alpha_0$ .

Let

$$\alpha := \alpha_0 + \frac{2r}{10}, \beta := \alpha_0 + \frac{8r}{10},$$

and let  $k_0$  be such that if  $k \geq k_0$ , then  $|c_{j_{r(k)}} - \alpha_0| < \frac{r}{10}$ . Assume  $k \geq k_0$ ; then  $c_{j_{r(k)}} \in [\alpha_0 - r/10, \alpha_0 + r/10]$ . Thus  $\alpha_0 - r/10 \leq c_{j_{r(k)}} \leq \alpha_0 + r/10 < \alpha < \beta$ , and therefore  $0 < \beta - c_{j_{r(k)}} \leq 9r/10$ . Since  $c_{j_{r(k)}+1} - c_{j_{r(k)}} \geq r$ , we conclude that  $\beta < c_{j_{r(k)}+1}$ ; thus  $[\alpha, \beta] \subset (c_{j_{r(k)}+1}, c_{j_{r(k)}})$ . Since  $c_{j_{r(k)}}$  and  $c_{j_{r(k)}+1}$  are consecutive  $\ell$ -zeros of  $T_{r(k)}$ , this implies that  $[\alpha, \beta]$  cannot contain an  $\ell$ -zero of  $T_{r(k)}$ . From Theorem 3(e) we therefore conclude that either  $[\alpha, \beta]$  contains one left endpoint of an equioscillation interval of  $T_{r(k)}$ , or  $[\alpha, \beta]$  contains no left endpoint of an equioscillation interval of  $T_{r(k)}$ . We will consider the first alternative: the proof of the second alternative is similar and will be omitted.

Assume that  $d_m^{r(k)} \in [\alpha, \beta]$ , and let  $D$  denote the set of the remaining left endpoints of equioscillation intervals of  $T_{r(k)}$ . Thus  $D \subset [a, \alpha] \cup [\beta, b]$  and  $D$  has  $r(k)$  elements. Choosing now  $\alpha < x_1 < x_2 < x_3 < x_4 < \beta$ , let  $f(x) \in C([a, b])$  be defined by

$$f(x) := \begin{cases} 0 & x \in [a, \alpha] \cup [\beta, b] \\ 2 & x = x_1, x = x_3 \\ -2 & x = x_2, x = x_4 \end{cases},$$

and by linear interpolation elsewhere in  $[a, b]$ .

Assume that for some  $n$  there is a function  $q \in S(G_n)$  such that  $\|f - q\| < 1/2$ . Let  $k \geq k_0$ ,  $r(k) \geq n$ , and let  $g$  be an element of best approximation from  $S(G_{r(k)})$  to  $f$ . Since  $S(G_n) \subset S(G_{r(k)})$ , we see that  $\|f - g\| \leq \|f - q\| < 1/2$ . The definition of  $f$  implies that

$$|g(d)| < 1/2, \quad d \in D.$$

We therefore conclude that

$$\text{sign}[T_{r(k)} - g](d) = \text{sign}T_n(d), \quad d \in D.$$

Moreover,  $\{x_1, x_2, x_3, x_4\}$  is a strong alternation for  $T_{r(k)} - g$  in  $(\alpha, \beta)$ . Selecting three of these four points appropriately and joining them to the set  $D$ , we see that  $T_{r(k)} - g$  has a strong alternation of length  $r(k) + 3$ . Since  $T_{r(k)} - g \in G_{r(k)}$ , and  $G_{r(k)}$  is a weak Markov system, and therefore cannot have a strong alternation of length larger than  $r(k) + 1$ , we have obtained a contradiction. This shows that (a)  $\Rightarrow$  (b).

To prove the converse let us assume, as in [10, Theorem 3], that  $S(G)$  is not dense. Then there is a nonzero Borel measure  $\mu$  such that for every function  $g \in S(G)$

$$\int_a^b g(t) d\mu(t) = 0.$$

Let  $\varepsilon > 0$ ,  $b_2$ , be such that  $a_1 < b_2 < b_1$  and  $a_2$  such that  $a_1 < a_2 < b_2$  and  $\mu([a_2, b_2]) < \mu([a, b])/6$ . The proof now is the same as in [10, Theorem 3, (b)  $\Rightarrow$  (a)], with  $a_2, b_2$  instead of  $a_1, b_1$ , and Lemma 1 replacing [10, Lemma 2] because, since  $[a, a_1] \cup [b_1, b] \subset [a, a_2] \cup [b_2, b]$ , Lemma 1 is applicable.  $\square$

**Corollary 1.** *If  $\lim_{n \rightarrow \infty} M_n = 0$  for one associated family of Tchebychev polynomials, then  $\lim_{n \rightarrow \infty} M_n = 0$  for every associated family of Tchebychev polynomials.*

## 5. JACKSON TYPE THEOREMS FOR FINITE SYSTEMS

In this section we will assume that  $G_n \subset C[a, b]$  is a normalized weak Markov system in  $[a, b]$ . Let  $E_n(f)$  denote the distance from  $f$  to  $S(G_n)$  in the norm of the supremum, and let  $\underline{a} := a - 1$  and  $\underline{b} := b + 1$ .

For each  $g \in G_n$  let us define  $\bar{g}$  on  $[\underline{a}, \underline{b}]$  as follows:

$$\bar{g}(x) := \begin{cases} g(a) & \text{if } \underline{a} \leq x \leq a \\ g(x) & \text{if } a < x < b \\ g(b) & \text{if } b \leq x \leq \underline{b} \end{cases}.$$

It is clear that  $\overline{G_n} := \{\bar{g}_0, \dots, \bar{g}_n\}$  is a normalized weak Markov system on  $[\underline{a}, \underline{b}]$ .

Let  $L(s) := \frac{1}{\sqrt{2\pi}} e^{-s^2/2}$  and  $L_k(s) := kL(ks)$ ;  $k \geq 1$ . For every  $f \in C[\underline{a}, \underline{b}]$  we set  $f^{(k)} := f * L_k$ , i. e.

$$f^{(k)}(x) = \int_{\underline{a}}^{\underline{b}} f(s) L_k(x-s) ds = \int_{-\infty}^{\infty} f(s) L_k(x-s) ds,$$

where in the second integral we understand  $f$  to equal 0 outside the interval  $[\underline{a}, \underline{b}]$ .

Under these conditions,  $\lim_{k \rightarrow \infty} f^{(k)} = f$ , uniformly on every closed subinterval of  $(\underline{a}, \underline{b})$ , and, if  $d = \underline{a}$ , or  $d = \underline{b}$ , then  $f^{(k)}(d)$  converges to  $\frac{1}{2}f(d)$ . Moreover,  $\overline{G_n^{(k)}} := \{\bar{g}_0^{(k)}, \dots, \bar{g}_n^{(k)}\}$  is an ECT system on  $[\underline{a}, \underline{b}]$  [13, pag. 15]. In particular, the functions in  $\overline{G_n^{(k)}}$  are an ECT system on  $[a, b]$ , and  $\bar{g}^{(k)}$  converges uniformly to  $\bar{g}$  on  $[a, b]$ , for every  $g \in S(G_n)$ .

Let  $g^{(k)}$  denote the restriction to the interval  $[a, b]$  of  $\bar{g}^{(k)} \in S(\overline{G_n^{(k)}})$ , and let  $G_n^{(k)} := \{g_0^{(k)}, \dots, g_n^{(k)}\}$ . Each  $g \in S(G_n)$  is in one-to-one correspondence with  $\bar{g} \in S(\overline{G_n})$ , which (for each  $k$ ) is in one-to-one correspondence with  $\bar{g}^{(k)} \in S(\overline{G_n^{(k)}})$ , which in turn are in one-to-one correspondence with its restrictions  $g^{(k)}$ :

$$g \longleftrightarrow \bar{g} \longleftrightarrow \bar{g}^{(k)} \longleftrightarrow g^{(k)}.$$

However, it is clear that  $g^{(k)} \neq g * L_k$ .

We now need a slight generalization of the main result of [11]. The proof is similar.

**Lemma 2.** *Let  $f \in C[a, b]$ , and let  $\{f_k\} \subset C[a, b]$  be a sequence that converges uniformly to  $f$  in  $[a, b]$ . For each  $k \geq 1$  let  $m_k$  be the element of best approximation to  $f_k$  from  $S(G_n^{(k)})$ . Then there is a subsequence  $\{k_j\}$  such that  $\{m_{k_j}\}$  converges uniformly in  $[a, b]$  to an element of best approximation  $m$  to  $f$  from  $S(G_n)$ . Moreover,  $f - m$  has an equioscillation.*

*Proof.* Since  $\|f_k - m_k\| \leq \|f_k - 0\|$ , and therefore  $\|m_k\| \leq 2\|f_k\|$ , we see that  $\{m_k\}$  is uniformly bounded. Since  $m_k = \sum_{i=0}^n \alpha_i^k g_i^{(k)}$ , there is a subsequence  $\{k_{1,j}\}$  and numbers  $\alpha_0, \dots, \alpha_n$ , such that  $\alpha_i^k \rightarrow \alpha_i$ ,  $i = 0, \dots, n$ , if  $k = k_{1,j} \rightarrow \infty$ .

Let  $m = \sum_{i=0}^n \alpha_i g_i$ , and  $g \in S(G_n)$ . Since  $m \in S(G_n)$ , we see that  $f_k - g^{(k)} \rightarrow f - g$  and  $f_k - m_k \rightarrow f - m$  uniformly in  $[a, b]$ . Since  $\|f_k - m_k\| \leq \|f_k - g^{(k)}\|$  we conclude that

$$\|f - m\| \leq \|f - g\|,$$

i. e.,  $m$  is an element of best approximation to  $f$ .

Moreover, if  $a \leq x_0^k < \dots < x_{n+1}^k \leq b$  is an equioscillation for  $f_k - m_k$ , then there is a subsequence  $\{k_{2,j}\}$  of  $\{k_{1,j}\}$ , a constant  $\varepsilon = \pm 1$ , and points  $a \leq x_0, \dots, x_{n+1} \leq b$ , such that if  $k = k_{2,j}$ , then

$$[f_k - m_k](x_i^k) = \varepsilon(-1)^i \|f_k - m_k\|,$$

and  $\lim_{j \rightarrow \infty} x_i^k \rightarrow x_i$ ,  $0 \leq i \leq n+1$ . Thus

$$[f - m](x_i) = \varepsilon(-1)^i \|f - m\|, \quad \varepsilon = 1 \text{ or } \varepsilon = -1.$$

This implies that  $[f - m](x_i) = -[f - m](x_{i+1}) \neq 0$ ,  $0 \leq i \leq n$ , and therefore that the points  $x_i$  are all different.  $\square$

For each  $k \geq 1$  and  $a \leq x_0 < \dots < x_{n+1} \leq b$ , let

$$\mathbf{D}^{(k)} := \{g_i^{(k)}(x_{j+1}) - g_i^{(k)}(x_j) : 1 \leq i \leq n; 0 \leq j \leq n\},$$

let  $\mathbf{D}_j^{(k)}$  be obtained from  $\mathbf{D}$  by deleting the  $j^{\text{th}}$  column, and  $d_j^{(k)} := \det \mathbf{D}_j^{(k)}$ , for  $0 \leq j \leq n$ .

In [10, Lemma 5] we showed that  $d_j^{(k)} \geq 0$  and  $\sum_{j=0}^n d_j^{(k)} > 0$ . Setting

$$a_j^{(k)} := \frac{d_j^{(k)}}{2 \sum_{j=0}^n d_j^{(k)}}$$

and

$$\delta^{(k)} := \sup_{a \leq x_0 < \dots < x_{n+1} \leq b} \left\{ \sum_{j=0}^n a_j^{(k)} [g_1^{(k)}(x_{j+1}) - g_1^{(k)}(x_j)] \right\},$$

then, if  $\omega(f)$  denotes the modulus of continuity of  $f$ , we have:

**Theorem 6.** *Let  $f \in C[a, b]$  and  $\delta := \overline{\lim}_{k \rightarrow \infty} \delta^{(k)}$ . Then  $\delta < \infty$  and*

$$E_n(f) \leq \frac{3}{2} \omega(f \circ g_1^{-1}; \delta).$$

*Proof.* Since  $|a_j^{(k)}| \leq \frac{1}{2}$  and  $\|g_1^{(k)}\| \leq \|g_1\|$  for every  $k$ , we see that the sequence  $\{\delta^{(k)}\}$  is bounded; thus  $\delta = \overline{\lim}_{k \rightarrow \infty} \delta^{(k)} < \infty$ . Let  $m_k$  be the element of best approximation to  $f$  from  $S(G_n^{(k)})$ . Since  $\{f^{(k)}\}$  converges to  $f$ , uniformly in  $[a, b]$ , applying Lemma 2 we obtain a subsequence  $\{k_j\}$  and an element of best approximation  $m$ , to  $f$  from  $S(G_n)$ , such that  $\{m_{k_j}\}$  converges uniformly to  $m$  in  $[a, b]$ . Now, let us choose a subsequence  $\{k_{1,j}\}$  of  $\{k_j\}$  such that  $\delta^{(k_{1,j})} \rightarrow \delta$  when  $j \rightarrow \infty$ . For convenience, let us denote it again by  $\{k\}$ .

Let  $f_1 := f \circ g_1^{-1}$ ,  $\varepsilon > 0$ , and let  $r_0 > 0$  be such that if  $r < r_0$ , then  $\omega(f_1; r) < \varepsilon/2$ . Choosing  $k_0$  so that if  $k \geq k_0$  then  $|\delta^{(k)} - \delta| < r_0/2$ , we have:

$$\omega(f_1; \delta^{(k)}) \leq \omega(f_1; \delta) + \omega(f_1; r_0/2) < \omega(f_1; \delta) + \varepsilon/2$$

and

$$\omega(f_1; \delta) \leq \omega(f_1; \delta^{(k)}) + \omega(f_1; r_0/2) < \omega(f_1; \delta^{(k)}) + \varepsilon/2.$$

Thus  $\omega(f \circ g_1^{-1}; \delta^{(k)}) \rightarrow \omega(f \circ g_1^{-1}; \delta)$  as  $k \rightarrow \infty$ .

Applying [10, Theorem 6] to  $G_n^{(k)}$  we have:

$$E_n^{(k)}(f) := \|f - m_k\| \leq \frac{3}{2} \omega(f \circ g_1^{-1}; \delta^{(k)}).$$

Making  $k \rightarrow \infty$ , the assertion follows.  $\square$

For each  $t \in [a, b]$  and  $k \geq 1$ , let  $\sigma_{k,t}$  be the function defined in  $[a, b]$  as follows:

$$\sigma_{k,t}(s) = \begin{cases} 0 & \text{for } a \leq s \leq t \\ g_1^{(k)}(s) - g_1^{(k)}(t) & \text{for } t < s \leq b \end{cases}.$$

If  $t$  is arbitrary but fixed, it is clear that as  $k \rightarrow \infty$  the function  $\sigma_{k,t}$  converges uniformly on  $[a, b]$  to

$$\sigma_t(s) = \begin{cases} 0 & \text{for } a \leq s \leq t \\ g_1(s) - g_1(t) & \text{for } t < s \leq b \end{cases}.$$

Let  $\theta_{k,t}$  the element of best approximation to  $\sigma_{k,t}$  from  $S(G_n^{(k)})$ , and let  $E_n^{(k)}(\sigma_{k,t}) := \|\sigma_{k,t} - \theta_{k,t}\|$ .

**Lemma 3.** *Let  $n \geq 0$  be arbitrary but fixed. The sequence  $\{E_n^{(k)}(\sigma_{k,t}) : k \geq 1\}$ , of functions of  $t$ , is uniformly bounded and uniformly continuous.*

*Proof.* The uniform boundedness follows from

$$|E_n^{(k)}(\sigma_{k,t})| = \|\sigma_{k,t} - \theta_{k,t}\| \leq \|\sigma_{k,t} - 0\| \leq 2 \|g_1^{(k)}\| \leq 2 \|g_1\|$$

If  $t_1, t_2 \in [a, b]$ , then

$$\|\sigma_{k,t_1} - \theta_{k,t_1}\| \leq \|\sigma_{k,t_1} - \theta_{k,t_2}\| \leq \|\sigma_{k,t_1} - \sigma_{k,t_2}\| + \|\sigma_{k,t_2} - \theta_{k,t_2}\|.$$

Therefore

$$E_n^{(k)}(\sigma_{k,t_1}) - E_n^{(k)}(\sigma_{k,t_2}) \leq \|\sigma_{k,t_1} - \sigma_{k,t_2}\|.$$

A similar argument yields

$$E_n^{(k)}(\sigma_{k,t_2}) - E_n^{(k)}(\sigma_{k,t_1}) \leq \|\sigma_{k,t_2} - \sigma_{k,t_1}\|.$$

Thus

$$|E_n^{(k)}(\sigma_{k,t_2}) - E_n^{(k)}(\sigma_{k,t_1})| \leq \|\sigma_{k,t_2} - \sigma_{k,t_1}\|.$$

But  $\|\sigma_{k,t_2} - \sigma_{k,t_1}\| \leq |g_1^{(k)}(t_2) - g_1^{(k)}(t_1)|$  (cf. [10, Lemma 6]). However,

$$|g_1^{(k)}(t_2) - g_1^{(k)}(t_1)| \leq \int_{-\infty}^{\infty} |g_1(t_2 - s) - g_1(t_1 - s)| L_k(s) ds,$$

which implies that the sequence  $\{g_1^{(k)} : k \geq 1\}$  is uniformly continuous, which in turn implies that  $\{E_n^{(k)}(\sigma_{k,t}) : k \geq 1\}$  is uniformly continuous.  $\square$

**Theorem 7.** *Let  $G_n \subset C[a, b]$  be a normalized weak Markov system in  $[a, b]$ . Then*

- (a)  $E_n(\sigma_t)$  is a continuous function of  $t$ .
- (b) If

$$\Delta_n := \max_{a \leq t \leq b} E_n(\sigma_t),$$

then

$$\delta \leq \sqrt{[g_1(b) - g_1(a)] \Delta_n}.$$

*Proof.* Applying Lemma 3 and Arzelà's theorem we see that there is a sequence  $\{k_{1,j}\}$  such that  $E_n^{(k_{1,j})}(\sigma_{k_{1,j},t})$  converges uniformly on  $[a, b]$  to a continuous function  $E$ .

For each fixed  $t \in [a, b]$ , Lemma 2. implies that there is a subsequence  $\{k_{2,j}\}$  of  $\{k_{1,j}\}$  such that  $E_n^{(k_{2,j})}(\sigma_{k_{2,j},t})$  converges to  $E_n(\sigma_t)$ . This shows that  $E(t) = E_n(\sigma_t)$ , and (a) follows.

Setting  $k = k_{2,j}$  and  $\Delta_n^{(k)} := \max_{a \leq t \leq b} E_n^{(k)}(\sigma_{k,t})$ , we see that  $\Delta_n^{(k)} \rightarrow_{k \rightarrow \infty} \Delta_n$ . Applying [10, Theorem 7] to  $G_n^{(k)}$ , we have:

$$\delta^{(k)} \leq \sqrt{[g_1^{(k)}(b) - g_1^{(k)}(a)] \Delta_n^{(k)}}.$$

Finally, if  $k = k_{3,j}$  is a subsequence of  $k_{2,j}$  such that  $\lim_{k \rightarrow \infty} \delta^{(k)} = \delta$ , we have:

$$\delta = \lim_{k \rightarrow \infty} \delta^{(k)} \leq \lim_{k \rightarrow \infty} \sqrt{[g_1^{(k)}(b) - g_1^{(k)}(a)] \Delta_n^{(k)}} = \sqrt{[g_1(b) - g_1(a)] \Delta_n}.$$

$\square$



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# Interpolating Scaling Functions with Duals

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## Abstract

This paper is concerned with the construction of smooth dual functions for a given family of interpolating scaling functions. The construction is based on a combination of the results in [2] and [11]. Several examples of dual functions are presented, including a continuously differentiable dual basis for the quincunx matrix.

**Key Words:** Interpolating scaling functions, Strang-Fix-conditions, biorthogonal wavelet bases, expanding scaling matrices, dual functions, Hölder regularity.

**AMS Subject classification:** 41A05, 42C40, 41A30, 41A63

## 1 Introduction

The construction of multivariate wavelets and scaling functions has been a field of increasing importance over the last years. A large variety of different construction principles has been published for orthogonal wavelets, biorthogonal wavelets, wavelets on spheres, scaling functions on general bounded and unbounded manifolds, scaling functions for specific operators (Radon transform, pseudo-differential operators, vaguelette bases) and many more.

Wavelets are usually constructed by means of a so-called scaling function. In general, a function  $\phi \in L_2(\mathbf{R}^d)$  is called a *scaling* function or a *refinable* function if it satisfies a *two-scale-relation*

$$\phi(x) = \sum_{k \in \mathbf{Z}^d} a_k \phi(Ax - k), \quad \mathbf{a} = \{a_k\}_{k \in \mathbf{Z}^d} \in \ell_2(\mathbf{Z}^d), \quad (1.1)$$

where  $A$  is an *expanding* integer scaling matrix, i.e., all its eigenvalues have modulus larger than one.

Current interest centers around the construction of multivariate interpolating scaling functions  $\phi$ , see e.g. [2, 3, 5, 6, 7, 8, 14], i.e. in addition to (1.1) one requires that  $\phi$  is at least continuous and satisfies

$$\phi(k) = \delta_{0,k}, \quad k \in \mathbf{Z}^d. \quad (1.2)$$

Interpolating scaling functions are needed for various applications e.g. CAGD or collocation methods for operator equations. These applications also require some smoothness of the scaling function. This problem has been solved satisfactory for  $\phi$  itself, even for the notorious quincunx matrix.

The next step of the construction process asks to find a *dual* scaling function  $\tilde{\phi}$  which satisfies

$$\langle \phi(\cdot), \tilde{\phi}(\cdot - k) \rangle = \delta_{0,k}, \quad k \in \mathbf{Z}^d. \quad (1.3)$$

However the best result so far for the quincunx matrix yields a dual scaling function  $\tilde{\phi} \in C^\alpha$  with  $\alpha = 0.3132$ , see [11]. The aim of this paper is to construct duals for interpolating scaling functions which are continuously differentiable. In Section 5 a dual function  $\tilde{\phi} \in C^\alpha$  for the quincunx matrix with  $\alpha = 1.9528$  is constructed.

This result is based on a combination of three different techniques:

- construction of smooth interpolating multivariate scaling functions [2],
- construction of duals for interpolating scaling functions [11],
- estimating the regularity of scaling functions using the techniques of [15].

The construction of smooth dual functions is the cornerstone for further developments. Given such a dual function, there exist several ways to construct a *biorthogonal* wavelet basis, i.e., two sets  $\{\psi_i\}_{i \in I}$  and  $\{\tilde{\psi}_{i'}\}_{i' \in I}$  of functions satisfying

$$\langle |\det A|^{j/2} \psi_i(A^j \cdot - k), |\det A|^{j'/2} \tilde{\psi}_{i'}(A^{j'} \cdot - k') \rangle = \delta_{i,i'} \delta_{j,j'} \delta_{k,k'}, \quad (1.4)$$

see, e.g., [11] and [12] for details. Moreover, the existence of dual wavelets is essential for establishing characterizations of smoothness spaces such as Sobolev or Besov spaces. In fact, under certain regularity and approximation assumptions the existence of dual wavelets imply the equivalence of the Sobolev and Besov norms of a function to weighted sequence norms of its wavelet coefficients, see, e.g., [13] and [4] for details.

The construction of dual functions for interpolating scaling functions is a fairly recent research topic. First examples were obtained in [11]. This paper mainly deals with dual

scaling functions for the classical box splines associated with the usual dyadic dilation matrix. Furthermore, some results concerning the quincunx matrix  $A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$  are included.

The results in [11] are derived by convolving a given interpolating scaling function with a suitable distribution. This distribution does not have any smoothness, i.e. this operation clearly diminishes the regularity of the resulting dual function  $\tilde{\phi}$ .

Therefore the whole construction only works satisfactory when the primal function  $\phi$  is sufficiently smooth. Such a family of smooth interpolating scaling functions was constructed in [2].

Hence we apply the construction principle of [11] to the scaling functions constructed in [2], this leads to a new family of biorthogonal scaling functions for the quincunx matrix  $A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$  which has the advantage that the dual functions are much smoother when compared to the results in [11].

This paper is organized as follows. In Section 2, we briefly recall the basic setting of interpolating scaling functions. In Section 3, we explain the construction of [2] as far as it is needed for our purposes. Then, in Section 4, we recall the approach derived in [11]. Finally, in Section 5, we combine both approaches and present a detailed regularity analysis using the smoothness estimates of [15].

For later use, let us fix some notation. Let  $q = |\det A|$ . Furthermore, let  $R = \{\rho_0, \dots, \rho_{q-1}\}$ ,  $R^T = \{\tilde{\rho}_0, \dots, \tilde{\rho}_{q-1}\}$  denote complete sets of representatives of  $\mathbf{Z}^d/A\mathbf{Z}^d$  and  $\mathbf{Z}^d/B\mathbf{Z}^d$ ,  $B = A^T$ , respectively. Without loss of generality, we shall always assume that  $\rho_0 = \tilde{\rho}_0 = 0$ .

## 2 The Setting

In the sequel, we shall only consider compactly supported scaling functions, furthermore we shall always assume that  $\text{supp } \mathbf{a} := \{k \in \mathbf{Z}^d \mid a_k \neq 0\}$  is finite. Computing the Fourier transform of both sides of (1.1) yields

$$\hat{\phi}(\omega) = \sum_{k \in \mathbf{Z}^d} \frac{1}{q} a_k e^{-i\langle k, B^{-1}\omega \rangle} \hat{\phi}(B^{-1}\omega). \quad (2.1)$$

By iterating (2.1) we obtain

$$\hat{\phi}(\omega) = \prod_{j=1}^{\infty} a(e^{-iB^{-j}\omega}), \quad (2.2)$$

where the *symbol*  $a(z)$  is defined by

$$a(z) := \frac{1}{q} \sum_{k \in \mathbf{Z}^d} a_k z^k. \quad (2.3)$$

Here we use the notation  $z = z(\omega) = e^{-i\langle \cdot, \omega \rangle}$  and  $z^k$  is the short hand notation for  $e^{-i\langle k, \omega \rangle}$ . We will mainly use the  $z$ -notation in this paper, i.e.  $a(1)$  refers to the value of

the symbol at  $\omega_1 = \dots \omega_d = 0$ . It will be stated explicitly, whenever we go back to the  $\omega$ -notation.

All known procedures for constructing multivariate scaling functions start with a symbol  $a(z)$ , which by Equation (2.2) determines  $\phi$ . Then the question arises which conditions on  $a(z)$  guarantee that  $\phi$  according to (2.2) is well-defined in  $L_2(\mathbf{R}^d)$  and has some additional desirable properties such as sufficient smoothness. Moreover, for our purposes, we have to clarify how the interpolating property (1.2) can be guaranteed. The following two conditions are necessary:

$$(C1) \quad a(1) = 1;$$

$$(C2) \quad \sum_{\tilde{\rho} \in R^T} a(\zeta_{\tilde{\rho}} e^{-iB^{-1}\omega}) = 1, \quad \text{where} \quad \zeta_{\tilde{\rho}} := e^{-2\pi i B^{-1}\tilde{\rho}}.$$

The following condition is not necessary, but it can be easily established in many cases and it is required for the construction of [11] as well as for the regularity estimates in Section 5. Moreover this condition already implies that the resulting scaling function is at least continuous:

$$(C3) \quad a(z) \geq 0 \quad .$$

Usually, conditions (C1)–(C3) are the starting point for the construction of a suitable symbol and the related interpolatory scaling function. Nevertheless, we want to point out that they are not sufficient in general.

Several procedures are known for constructing interpolating scaling functions, however the true challenge asks for constructing smooth scaling functions. To this end, one often requires that the *Strang-Fix-conditions* of order  $N$  are satisfied, i.e.,

$$(C4) \quad \left( \frac{\partial}{\partial \omega} \right)^l a(2\pi B^{-1}\tilde{\rho}) = 0 \quad \text{for all} \quad |l| \leq N \quad \text{and all} \quad \tilde{\rho} \in R^T \setminus \{0\}.$$

This paper is concerned with the construction of pairs of biorthogonal functions  $(\phi, \tilde{\phi})$  where  $\phi$  is an interpolating scaling function and the dual scaling function  $\tilde{\phi}$  satisfies (1.3). A necessary condition for the symbol  $\tilde{a}$  of the dual scaling function  $\tilde{\phi}$  in order to satisfy (1.3) is given by

$$1 = \sum_{\tilde{\rho} \in R^T} a(\zeta_{\tilde{\rho}} z) \overline{\tilde{a}(\zeta_{\tilde{\rho}} z)}. \quad (2.4)$$

Therefore the usual way to find a dual function for a given scaling function is to construct a symbol  $\tilde{a}(z)$  satisfying (2.4) and to check that the corresponding refinable function exists in  $L_2$  and is sufficiently regular. Indeed, we measure the success of a construction method for the dual function by the achievable Hölder regularity of  $\tilde{\phi}$ .

### 3 Smooth Interpolating Scaling Functions

As outlined in the introduction our search for smooth dual functions  $\tilde{\phi}$  requires a smooth interpolating scaling function  $\phi$ . The details on how to construct a suitable  $\tilde{\phi}$ , resp.  $\tilde{a}$ , for a given  $\phi$ , resp.  $a$  are outlined in Section 4.

First of all we briefly recall the construction of interpolating scaling functions developed in [2]. It is based on Lagrange interpolation and can be interpreted as a generalization of the univariate approach derived in [10] to the multivariate situation.

We say that a symbol  $a(z)$  satisfies the *Strang–Fix conditions* with respect to a set of polynomials  $\Pi$ , if  $(D = \frac{\partial}{\partial \omega})$

$$(p(D)a)(2\pi B^{-1}\tilde{\rho}) = 0 \quad \text{for all } p \in \Pi, \tilde{\rho} \in R^T \setminus \{0\}. \quad (3.1)$$

For any subset  $\mathcal{T} \subseteq \mathbf{Z}^d$ ,  $\Pi_{\mathcal{T}}$  will always denote a finite-dimensional subspace of polynomials such that the Lagrange interpolation problem with respect to  $\mathcal{T}$  is uniquely solvable. Under this hypothesis the following theorem holds.

**Theorem 3.1** *Let  $\mathcal{P}$  be a subspace of  $\Pi_{\mathcal{T}}$  satisfying*

- (1) *If  $p \in \mathcal{P}$ , then  $p(c(Ax + \rho)) \in \Pi_{\mathcal{T}}$  for  $c \in \mathbf{C}$ ,  $\rho \in R$ ;*
- (2)  *$p(0) = 0$  for all  $p \in \mathcal{P}$ .*

*Then the symbol  $a(\omega)$  defined by*

$$a(\omega) = \frac{1}{q} + \frac{1}{q} \sum_{k \in \mathcal{T}} \sum_{\rho \in R \setminus \{0\}} p_k(-A^{-1}\rho) e^{-i\langle Ak + \rho, \omega \rangle} \quad (3.2)$$

*satisfies (C1), (C2), and the Strang–Fix conditions (3.1) with respect to  $\mathcal{P}$ .*

Since Lagrange interpolation on general sets of nodes in  $\mathbf{R}^d$  is far from understood, we restrict ourselves to very simple sets with additional symmetry. Let  $\mathcal{T}$  consist of all lattice points in a cube in  $\mathbf{R}^d$ , i.e., for  $N \in \mathbf{N}$  and  $\beta \in \mathbf{Z}^d$  we set

$$\mathcal{T} = \mathcal{T}_{L,\beta} := \{k \in \mathbf{Z}^d : \beta_i \leq k_i \leq N + \beta_i, \quad i = 1, \dots, d\} = (\beta + [0, N]^d) \cap \mathbf{Z}^d. \quad (3.3)$$

The Lagrange interpolation problem is always unsolvable on  $\mathcal{T}$  by the polynomial subspace

$$\Pi_{\mathcal{T}} = \text{span}\{x^k, k \in \mathbf{Z}^d, \|k\|_{\infty} \leq N, k_i \geq 0, i = 1, \dots, d\}. \quad (3.4)$$

The fundamental Lagrange interpolants are simply tensor products of the univariate Lagrange polynomials and can be written explicitly as

$$p_k(x) = \ell_{k_1}(x_1) \ell_{k_2}(x_2) \cdots \ell_{k_d}(x_d), \quad \ell_{k_i}(x_i) := \prod_{n=a_i, n \neq k_i}^{L+a_i} \frac{x_i - n}{k_i - n}. \quad (3.5)$$

This leads to the following corollary.

**Corollary 3.1** *Let  $\mathcal{T}$  and  $\Pi_{\mathcal{T}}$  be defined by (3.3) and (3.4), respectively. Then  $a(\omega)$  defined by (3.2) satisfies the Strang–Fix conditions with respect to  $\Pi_{\mathcal{T}}$ . In particular, the usual Strang–Fix conditions of order  $N$  are satisfied.*

It has been shown in [2] that under certain symmetry assumptions on the mask the resulting symbol is in fact real which is clearly necessary to ensure condition (C3). Moreover, in [2], this setting has been applied to the quincunx matrix  $A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ . Then  $q = 2$  and a set of representatives is given by  $\rho_0 = 0, \rho_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ . Moreover,  $-A^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1/2 \\ 1/2 \end{pmatrix}$  and  $\mathcal{T}$  needs to be symmetric about  $(-1/2, 1/2)$ . This is the case for  $\mathcal{T} = [-L, L-1] \times [-L+1, L] \cap \mathbf{Z}^2$ . Let  $\ell_n$  denote the basic Lagrange interpolation polynomial for  $n \in \{-L, -L+1, \dots, L-1\}$ . With

$$q_L(x) := \sum_{n=-L}^{L-1} \ell_n(-1/2) e^{-inx}, \quad (3.6)$$

we obtain for  $a(\omega)$  corresponding to (3.2)

$$a(\omega) = \frac{1}{2} + \frac{1}{2} e^{-i(\omega_1 + \omega_2)/2} q_L(\omega_1 + \omega_2) e^{-i(\omega_2 - \omega_1)/2} q_L(\omega_2 - \omega_1). \quad (3.7)$$

By construction, this symbol satisfies (C1) and (C2). Moreover, it has been shown that for any  $L$  condition (C3) is also satisfied and that the symbol indeed gives rise to an interpolating scaling function.

As an example, for  $L = 2$  the nonvanishing coefficients can be computed as follows.

$$\begin{aligned} a_{(0,0)} &= \frac{1}{2}; \\ a_{(1,0)} &= a_{(0,1)} = a_{(-1,0)} = a_{(0,-1)} = \frac{81}{512}; \\ a_{(3,0)} &= a_{(0,3)} = a_{(-3,0)} = a_{(0,-3)} = \frac{1}{512}; \\ a_{(2,1)} &= a_{(1,2)} = a_{(-1,2)} = a_{(-2,1)} = a_{(-2,-1)} = a_{(-1,-2)} = a_{(1,-2)} = a_{(2,-1)} = -\frac{9}{512}. \end{aligned} \quad (3.8)$$

## 4 Construction of Dual Functions

In this section, we briefly recall the algorithm for constructing a dual basis for a given interpolating scaling function as developed in [11]. The main result in [11] is a lifting scheme, which allows to construct a second smoother interpolating function from a given one.

Defining

$$b_{\tilde{\rho}}(z) = a(\zeta_{\tilde{\rho}} z), \quad \tilde{\rho} \in R^T, \quad (4.1)$$

condition (C2) may equivalently be written as

$$1 = \sum_{\tilde{\rho} \in R^T} b_{\tilde{\rho}}(z). \quad (4.2)$$



Hence, for any integer  $K$ ,

$$\left( \sum_{\tilde{\rho} \in R^T} b_{\tilde{\rho}}(z) \right)^{Kq} = \sum_{|\gamma|=qK} \left( C_{qK}^\gamma \prod_{\hat{\rho} \in R^T} b_{\hat{\rho}}^{\gamma_{\hat{\rho}}}(z) \right) = 1. \quad (4.3)$$

Here  $\gamma$  denotes a vector of dimension  $q$ , the coefficients of  $\gamma$  are indexed by  $\tilde{\rho} \in R^T = \{\tilde{\rho}_0, \dots, \tilde{\rho}_{q-1}\}$ .

By using (4.3), the following theorem was established in [11].

**Theorem 4.1** *Let  $a(z)$  be a symbol satisfying (4.2) for a dilation matrix  $A$  with  $q = |\det A|$ . Define*

$$\begin{aligned} G_0 &:= \left\{ \gamma \in \mathbf{N}_0^q : |\gamma| = qK, \gamma_0 > K \text{ and } \gamma_0 > \gamma_{\hat{\rho}}, \hat{\rho} \in R^T \setminus \{0\} \right\} \\ G_j &:= \left\{ \gamma \in \mathbf{N}_0^q : |\gamma| = qK, \gamma_0 > K \text{ and } \gamma_0 \geq \gamma_{\hat{\rho}}, \hat{\rho} \in R^T \setminus \{0\}, \text{ with exactly } j \text{ equalities} \right\}, \\ &\quad j = 1, \dots, q-2, \end{aligned}$$

and define

$$H_K := \sum_{j=0}^{q-2} \frac{1}{j+1} \left( \sum_{\gamma \in G_j} C_{qK}^\gamma a(z)^{\gamma_0-1} \prod_{\hat{\rho} \in R^T \setminus \{0\}} b_{\hat{\rho}}^{\gamma_{\hat{\rho}}}(z) \right) + C_{qK}^{(K, \dots, K)} \prod_{\hat{\rho} \in R^T} b_{\hat{\rho}}^K(z),$$

where  $C_{qK}^\gamma$  are the multinomial coefficients. Then the symbol  $a(z)H_K(z)$  also satisfies (4.2).

It can be checked that the symbol  $H_K$  can be factored as

$$H_K(z) = a(z)^K T_K(z) \quad (4.4)$$

for some suitable symbol  $T_K(z)$ . Consequently, the refinable function associated with  $a(z)H_K(z)$  is obtained by convolving the original function  $K-1$ -times with itself followed by a convolution with some distribution. Since  $a(z)H_K(z)$  satisfies (4.2), it is a candidate for a symbol corresponding to an interpolating scaling function. Indeed, the following corollary was established in [11].

**Corollary 4.1** *Let  $a(z)$  be the symbol of a continuous compactly supported interpolating refinable function and assume that  $a(z)$  satisfies (C3). If the refinable function corresponding to  $a(z)H_K(z)$  is continuous, then it is interpolating.*

This approach can now be used to construct dual functions for the given interpolating scaling function  $\phi$ . Indeed, by recalling the necessary condition (2.4), we see that by Theorem 4.1

$$\tilde{a}(z) := \overline{H_K(z)} = \overline{a(z)^K T_K(z)} \quad (4.5)$$

is a natural candidate for a symbol associated with a dual function. The following corollary is again taken from [11].

**Corollary 4.2** *If the refinable function corresponding to the mask  $H_K$  is in  $L_2(\mathbf{R}^d)$ , then it is stable and dual to  $\phi$ .*

## 5 Smooth Dual Pairs on the Quincunx Grid

In this section, we want to employ the algorithm described in Section 4 to construct smooth dual pairs for the quincunx matrix  $A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ . Corollary 4.2 tells us how to proceed:

- Find a continuous interpolating refinable function  $\phi$ ;
- Compute  $H_K$  according to Theorem 4.1;
- Check that the corresponding refinable function is contained in  $L_2(\mathbf{R}^d)$ .

Clearly the last part is the most nontrivial step. Moreover, it is desirable to find dual functions which are as smooth as possible. We are therefore faced with the problem of estimating the regularity of a refinable function by only using the refinement mask. This problem has attracted several people in the last few years, see, e.g., [1, 9, 14, 15]. Let us briefly recall the basic ideas. We want to find

$$\alpha^* := \sup\{\alpha : \phi \in C^\alpha\}.$$

It is well-known that  $\alpha^* \geq \kappa_{\text{sup}}$ , where  $\kappa_{\text{sup}}$  is defined by

$$\kappa_{\text{sup}} := \sup\{\kappa : \int_{\mathbf{R}^d} (1 + |\omega|)^\kappa |\hat{\phi}(\omega)| d\omega < \infty\}. \quad (5.1)$$

Our aim is to estimate  $\kappa_{\text{sup}}$  from below. One typical result in this direction reads as follows.

**Theorem 5.1** *For an integer  $N$ , let*

$$V_N := \{v \in \ell_0(\mathbf{Z}^d) : \sum_{k \in \mathbf{Z}^d} p(k)v_k = 0, \text{ for all } p \in \Pi_N\},$$

where  $\Pi_N$  denotes the polynomials of total degree  $N$ . Assume that  $A$  is a dilation matrix with a complete set of orthonormal eigenvectors, let  $|\lambda_{\max}|$  denote the eigenvalue of  $A$  with the largest modulus. Let  $\Omega$  denote a subset of  $\mathbf{Z}^d$  s.t.  $\text{supp } \mathbf{a} \subseteq \Omega$  and  $V_N$  is invariant under the matrix

$$\mathcal{H} := [qa_{Ak-l}]_{k,l \in \Omega}.$$

Assume that the symbol  $a(z)$  according to (2.3) is non-negative and satisfies Strang-Fix conditions of order  $N$ . Let  $\varrho$  be the spectral radius of  $\mathcal{H}|_{V_N}$ . Then the exponent  $\kappa_{\text{sup}}$  satisfies

$$\kappa_{\text{sup}} \geq -\frac{\log(\varrho)}{\log(|\lambda_{\max}|)}. \quad (5.2)$$

As already stressed in Section 4, the approach in [11] actually consists of a convolution of the starting interpolating function  $\phi$  with itself followed by a convolution with a distribution. This distribution may be ugly so that it may diminish the regularity of

the resulting function significantly. Therefore the method in [11] will only perform satisfactory for a sufficiently smooth starting mask.

Hence we combine this construction procedure with the approach in [2], which produces interpolating functions with a small mask but with a high order of Strang–Fix conditions. Since the Strang–Fix conditions serve as indicators for some smoothness, there is good reason to expect that the resulting refinable functions are quite regular. Indeed, by using Theorem 5.1 we obtained for  $L = 2$  and  $L = 3$ , respectively

$$\phi_2 \in C^\alpha \quad \text{for all } \alpha < 1.5156 \quad \text{and} \quad \phi_3 \in C^\alpha \quad \text{for all } \alpha < 2.3035. \quad (5.3)$$

Therefore we decided to use these functions as starting points. The next step is to compute the symbols  $H_K$ . For the quincunx matrix, we clearly have  $q = 2$  and the first four symbols can be computed explicitly, for the definition of  $b_0$  and  $b_1$  see (4.1):

$$\begin{aligned} H_1 &= b_0(1 + 2b_1), \\ H_2 &= b_0^2(b_0 + 4b_1 + 6b_1^2), \\ H_3 &= b_0^3(b_0^2 + 6b_0b_1 + 15b_1^2 + 20b_1^3), \\ H_4 &= b_0^4(b_0^7 + 8b_0^6b_1 + 28b_0^5b_1^2 + 56b_0^4b_1^3 + 70b_0^4b_1^4). \end{aligned} \quad (5.4)$$

For details, we refer again to [11]. Given  $a(z)$ , the corresponding symbols  $H_1, \dots, H_4$  can be computed by symbolic software such as MAPLE.

As an example, for  $L = 2$  and  $K = 1$  we obtain a mask with 65 non-zero coefficients:

$$\begin{array}{lll} H_{1,(-6,0)} = -1/65536; & H_{1,(-5,-1)} = 9/32768; & H_{1,(-5,1)} = 9/32768; \\ H_{1,(-4,-2)} = -63/65536; & H_{1,(-4,0)} = -81/16384; & H_{1,(-4,2)} = -63/65536; \\ H_{1,(-3,-3)} = -41/16384; & H_{1,(-3,-1)} = 567/32768; & H_{1,(-3,0)} = 1/256; \\ H_{1,(-3,1)} = 567/32768; & H_{1,(-3,3)} = -41/16384 & H_{1,(-2-4)} = -63/65536; \\ H_{1,(-2,-2)} = 369/8192; & H_{1,(-2,-1)} = -9/256; & H_{1,(-2,0)} = -3969/65536; \\ H_{1,(-2,1)} = -9/256; & H_{1,(-2,2)} = 369/8192; & H_{1,(-2,4)} = -63/65536; \\ H_{1,(-1,-5)} = 9/32768; & H_{1,(-1-3)} = 567/32768; & H_{1,(-1,-2)} = -9/256; \\ H_{1,(1,-1)} = -2583/16384; & H_{1,(-1,0)} = 81/256; & H_{1,(-1,1)} = -2583/16384; \\ H_{1,(-1,2)} = -9/256; & H_{1,(-1,3)} = 567/32768; & H_{1,(-15)} = 9/32768; \\ H_{1,(0,-6)} = -1/65536; & H_{1,(0,-4)} = -81/16384; & H_{1,(0,-3)} = 1/256; \\ H_{1,(0,-2)} = -3969/65536; & H_{1,(0,-1)} = 81/256; & H_{1,(0,0)} = 6511/4096; \\ H_{1,(0,1)} = 81/256; & H_{1,(0,2)} = -3969/65536; & H_{1,(0,3)} = 1/256; \\ H_{1,(0,4)} = -81/16384; & H_{1,(0,6)} = -1/65536; & H_{1,(1,-5)} = 9/32768; \\ H_{1,(1,-3)} = 567/32768; & H_{1,(1,-2)} = -9/256; & H_{1,(1,-1)} = -2583/16384; \\ H_{1,(1,0)} = 81/256; & H_{1,(1,1)} = -2583/16384; & H_{1,(1,2)} = -9/256; \\ H_{1,(1,3)} = 567/32768; & H_{1,(1,5)} = 9/32768; & H_{1,(2,-4)} = -63/65536; \\ H_{1,(2,-2)} = 369/8192; & H_{1,(2,-1)} = -9/256; & H_{1,(2,0)} = -3969/65536; \\ H_{1,(2,1)} = -9/256; & H_{1,(2,2)} = 369/8192; & H_{1,(2,4)} = -63/65536; \\ H_{1,(3,-3)} = -41/16384; & H_{1,(3,-1)} = 567/32768; & H_{1,(3,0)} = 1/256; \\ H_{1,(3,1)} = 567/32768; & H_{1,(3,3)} = -41/16384; & H_{1,(4,-2)} = -63/65536; \\ H_{1,(4,0)} = -81/16384; & H_{1,(4,2)} = -63/65536; & H_{1,(5,-1)} = 9/32768; \\ H_{1,(5,1)} = 9/32768; & H_{1,(6,0)} = -1/65536. & \end{array} \quad (5.5)$$

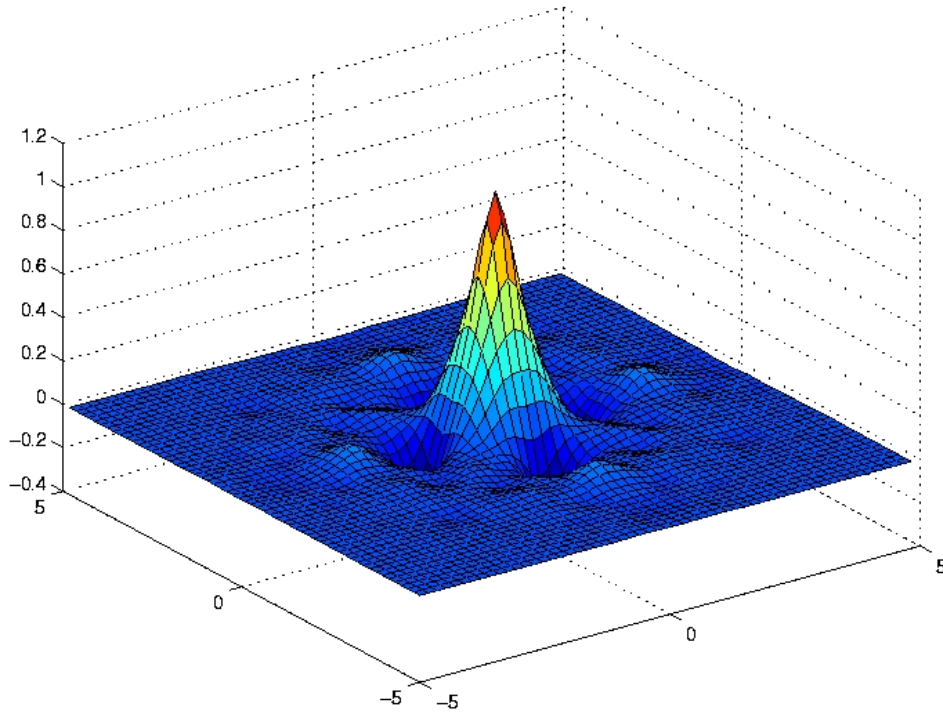


Figure 1: Visualization of the dual function for  $L = 3$ ,  $K = 2$ , this function satisfies  $\tilde{\phi} \in C^\alpha(\mathbf{R}^2)$  for  $\alpha = 1.9528$ .

We used Theorem 5.1 to estimate the regularity of the resulting refinable functions. The results are displayed in the following table.

$L$	$K$	$\kappa_{\text{sup}}$
2	1	-0.497
2	2	0.729
2	3	1.803
3	1	0.204
3	2	1.952

We see, that the regularity of the dual functions grows rapidly as  $K$  increases. For  $L = 2$ ,  $K = 1$  we do not get an  $L_2$ -function, but already the function with respect to  $L = 2$ ,  $K = 2$  is smoother than the smoothest one constructed in [11] which was contained in  $C^{0.313226}$ .

For  $L = 2$ ,  $K = 3$  the dual function is continuously differentiable. To our knowledge, examples for the quincunx matrix with these properties have not been constructed before. For  $L = 3$ ,  $K = 2$  the dual function is almost contained in  $C^2$ . It seems very likely that enlarging the values of  $N$  and  $K$  will produce even higher orders of regularity. However the computations become too time consuming, already the presented examples lead to eigenvalue problems for matrices with dimension  $> 4 * 10^3$ . This could only be handled with reasonable computer time by employing sparse matrix techniques.

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# On the Interference of the Weight and Boundary Contour for Orthogonal Polynomials over the Region

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## Abstract

In this work the estimation of the maximum norm of orthogonal polynomials over the region with respect to the weight is analyzed. It is observed that the norm of polynomials does not change for the conditions of weight and boundary curve.

*Keywords:* Orthogonal polynomials, Weight function, Quasircle, Conformal mapping, Quasiconformal mapping.

*Subject Classes:* 30A10, 30C10 41A17.

## 1 Introduction.

Let  $G$  be a finite region, with  $0 \in G$ , bounded by a Jordan curve  $L := \partial G$  and  $h(z)$  is a weight function in  $G$ . The polynomials  $\{K_n(z)\}$ ,  $\deg K_n = n$ ,  $n = 0, 1, 2, \dots$ , satisfying the condition

$$\iint_G h(z) K_n(z) \overline{K_m(z)} d\sigma_z = \delta_{n,m} \quad (1)$$

are called orthonormal polynomials for the pair  $(G, h)$ . They are determined uniquely if the coefficient of the term of the highest degree is positive.

These polynomials were first analyzed by Carleman [6]. He researched the Faber problem relating to generalization to simple connected region of Taylor series. Apart from these approximation problems, these polynomials were the subjects of many investigations who investigated asymptotical behaviors of the polynomials inside and closure of region  $G$ .

In the present study we have investigated the estimation problem of the maximum norm  $\|K_n\|_{C(\overline{G})} := \max\{|K_n(z)|, z \in \overline{G}\}$  of orthogonal polynomials over the region with respect to the weight. The polynomials are defined by the pair  $(G, h)$ . Therefore, variation of norm of these polynomials depends on the properties of the region  $G$  and weight  $h(z)$ . This dependency has been investigated for orthogonality along a curve at [7],[10],[11] and over the region at [1]-[3],[12].

## 2 Main definition and results.

Throughout this paper  $c, c_1, c_2, \dots$  are positive, and  $\varepsilon, \varepsilon_1, \varepsilon_2, \dots$  sufficiently small positive constants (in general, different in different relations), which depend on  $G$  in general. “ $a \prec b$ ” and “ $a \succ b$ ” are equivalent to  $a \leq b$  and  $c_1 a \leq b \leq c_2 a$  for some constants  $c, c_1, c_2$  respectively. Let  $w = \varphi(z)$  ( $w = \Phi(z)$ ) be the conformal mapping of  $G$  ( $\Omega := \overline{CG}$ ) onto the unit disc  $B := \{w : |w| < 1\}$  normalized by  $\varphi(0) = 0, \varphi'(0) > 0$  ( $\Phi(\infty) = \infty, \Phi'(\infty) > 0$ ) and let  $\psi := \varphi^{-1}$  ( $\Psi := \Phi^{-1}$ ).

**Definition 1** A bounded Jordan region  $G$  is called a  $k$ -quasidisk,  $0 \leq k < 1$ , if any conformal mapping  $\psi$  can be extended to a  $K$ -quasiconformal,  $K = \frac{1+k}{1-k}$ , homeomorphism of the plane  $\overline{C}$  on the  $\overline{C}$ . In that case the curve  $L := \partial G$  is called a  $K$ -quasicircle. The region  $G$  (curve  $L$ ) is called a quasidisk (quasicircle), if it is  $k$ -quasidisk ( $k$ -quasicircle) with some  $0 \leq k < 1$ .

**Theorem 2** Let  $G$  is a  $k$ -quasidisk for some  $0 \leq k < 1$ , and the weight function  $h(z)$  uniformly bounded away from zero, i.e.

$$h(z) \geq c > 0 \quad (2)$$

Then, for every  $n = 1, 2, \dots$

$$\|K_n\|_{C(\overline{G})} \leq c_1 n^{1+k}. \quad (3)$$

**Definition 3** We say that  $G \in Q_\alpha$ ,  $0 < \alpha \leq 1$ , if

- a)  $L$  is a quasicircle,
- b)  $\Phi \in Lip\alpha$ ,  $z \in \overline{\Omega}$ .



**Theorem 4** *Let  $G \in Q_\alpha$ , for some  $0 < \alpha \leq 1$  and  $h(z)$  satisfying the condition (2), Then, for every  $n = 1, 2, \dots$*

$$\|K_n\|_{C(\overline{G})} \leq c_2 \begin{cases} n^\delta, & \alpha < \frac{1}{2} \\ n^{\frac{1}{\alpha}}, & \alpha \geq \frac{1}{2} \end{cases} \quad (4)$$

where  $\delta = \delta(G)$ ,  $1 \leq \delta \leq 2$ , is a certain number.

Now, we assume that the function  $h(z)$  doesn't supply the condition (2). We define the function  $h(z)$  as the following:

$$h(z) = h_0(z) \prod_{i=1}^m |z - z_i|^{\gamma_i}, \quad (5)$$

where  $\{z_i\}$ ,  $i = \overline{1, m}$  is the fixed system of the points on  $L$ ;  $\gamma_i > -2$  and  $h_0(z)$  is satisfying the condition  $h_0(z) \geq c > 0$ .

**Definition 5** *We say that  $\Omega \in Q(\nu)$ ,  $0 < \nu < 1$ , if*

- i)  $L := \partial\Omega = \partial G$  is quasicircle,
- ii) *For  $\forall z \in L$ , there exists a  $r > 0$  and  $0 < \nu < 1$  such that a closed circular sector  $S(z; r, \nu) := \{\zeta : \zeta = z + re^{i\theta}, 0 \leq \theta_0 < \theta < \theta_0 + \nu\}$  of radius  $r$  and opening  $\nu\pi$  lies in  $\overline{G}$  with vertex at  $z$ .*

It is well known that each quasicircle satisfies the condition ii). Nevertheless, this condition imposed on  $L$  gives a new geometric characterization of the curve or region. For example, if the region  $G^*$  defined by

$$G^* := \left\{ z : z = re^{i\theta}, 0 < r < 1, \frac{\pi}{2} < \theta < 2\pi \right\},$$

then the coefficient of quasiconformality  $k$  of the  $G^*$  does not obtain so easily, whereas  $\Omega^* := CG^* \subset Q(\frac{1}{2})$ .

**Definition 6** *We say that  $\Omega \in Q_\alpha(\nu_1, \dots, \nu_m)$ ,  $0 < \nu_1, \dots, \nu_m < \alpha \leq 1$ , if there exist a system of points  $\{z_i\}$ ,  $i = \overline{1, m}$  on  $L$ , such that  $\Omega \in Q(\nu_i)$  for any points  $z_i \in L$  and  $\Phi \in \text{Lip}\alpha$ ,  $z \in \overline{\Omega} \setminus \{z_i\}$ .*

Assume that the system of points  $\{z_i\}$ ,  $i = \overline{1, m}$  mentioned in (5) and Definition 6 is identically ordered on  $L$ .

**Theorem 7** Let  $\Omega \in Q_\alpha(\nu_1, \dots, \nu_m)$ , for some  $0 < \nu_i < 1$  and  $\alpha(2 - \nu_i) \geq 1$ ,  $h(z)$  defined by (5) and, in addition

$$1 + \frac{\gamma_i}{2} = \frac{1}{\alpha(2 - \nu_i)} \quad (6)$$

is satisfied for any points  $z_i \in L$ ,  $i = \overline{1, m}$ . Then, for every  $n = 1, 2, \dots$

$$\|K_n\|_{C(\overline{G})} \leq c_3 n^{1/\alpha} \quad (7)$$

Comparing Theorem 7 with Theorem 4 we see that when equality (6) was satisfied, then the maximum norm of polynomials  $K_n(z)$  in  $\overline{G}$  acts itself identically, neither weight  $h(z)$  and boundary curve  $L$  have not got singularity nor they have got singularity. The equality given by (6) shows the interference condition of weight and boundary contour.

**Corollary 8** In the Definition 6, if the boundary arcs of the region  $\Omega$  joining the points  $\{z_j\} \in L$  are arcs of the class  $C(1, \alpha)$  then we can find the region with piecewise smooth boundary and having got in the joining points interior angles  $\nu_i\pi$ ,  $0 < \nu_i < 1$ . In this case the relation (6) and (7) will be given as follows:

$$1 + \frac{\gamma_i}{2} = \frac{1}{(2 - \nu_i)}, \quad i = \overline{1, m}, \quad (8)$$

$$\|K_n\|_{C(\overline{G})} \leq c_3 n. \quad (9)$$

This result extends the theorem of Suetin [12, Th.4.6] in case of  $0 < \nu_i < 1$ .

Let  $A_p(h, G)$ ,  $p > 0$  denotes the class of function  $f$  which analytic in  $G$  and satisfying the condition

$$\|f\|_{A_p} := \|f\|_{A_p(h, G)} := \left( \iint_G h(z) |f(z)|^p d\sigma_z \right)^{1/p} < \infty.$$

Since the polynomials  $K_n(z)$  have a minimal  $\|P_n\|_{A_p(h, G)}$ -norm in the class of all polynomials  $P_n(z)$ ,  $\deg P_n \leq n$ ,  $n = 1, 2, \dots$ , the Theorems 2-7 can be generalized for this class. In this case we have relation of the norms  $\|P_n\|_{C(\overline{G})}$  and  $\|P_n\|_{A_p(h, G)}$ .

**Theorem 9** Let  $P_n(z)$ ,  $\deg P_n \leq n$ ,  $n = 1, 2, \dots$ , is arbitrary polynomial and  $1 \leq p < \infty$ . Then

a) under the conditions of Theorem 2

$$\|P_n\|_{C(\overline{G})} \leq c_4 n^{\frac{2(1+k)}{p}} \|P_n\|_{A_p},$$

b) under the conditions of Theorem's 2 and 7

$$\|P_n\|_{C(\overline{G})} \leq c_5 n^{\frac{2}{p\alpha}} \|P_n\|_{A_p},$$

This estimations are sharp as exponent in the class of all polynomials of degree at most  $n$ .

### 3 Some auxiliary results

Let  $G$  is a quasidisk. Then there exists a quasiconformal reflection  $y(\cdot)$  across  $L$  such that  $y(G) = \Omega$ ,  $y(\Omega) = G$  and  $y(\cdot)$  fixes the points of  $L$ . The quasiconformal reflection  $y(\cdot)$  is such that it satisfied the following condition [5, p.26]

$$\begin{aligned} |y(\zeta) - z| &\asymp |\zeta - z|, \quad z \in L, \quad \varepsilon < |\zeta| < \frac{1}{\varepsilon} \\ |y_{\overline{\zeta}}| &\asymp |y_{\zeta}| \asymp 1, \quad \varepsilon < |\zeta| < \frac{1}{\varepsilon} \\ |y_{\overline{\zeta}}| &\asymp |y(\zeta)|^2, \quad |\zeta| < \varepsilon, \quad |y_{\overline{\zeta}}| \asymp |\zeta|^{-2}, \quad |\zeta| > \frac{1}{\varepsilon}. \end{aligned} \tag{10}$$

For  $t > 0$ , let  $L_t := \{z : |\varphi(z)| = t, \text{ if } t < 1, |\Phi(z)| = t, \text{ if } t > 1\}$ ,  $G_t := \text{int}L_t$ ,  $\Omega_t := \text{ext}L_t$ .

**Lemma 10** [3] *Let  $G$  be a quasidisk,  $z_1 \in L$ ,  $z_2, z_3 \in \Omega \cap \{z : |z - z_1| \prec d(z_1, L_{r_0})\}$ ;  $w_j = \Phi(z_j)$ ,  $j = 1, 2, 3$ . Then*

a) *The statements  $|z_1 - z_2| \prec |z_1 - z_3|$  and  $|w_1 - w_2| \prec |w_1 - w_3|$  are equivalent. So are  $|z_1 - z_2| \asymp |z_1 - z_3|$  and  $|w_1 - w_2| \asymp |w_1 - w_3|$ .*

b) *If  $|z_1 - z_2| \prec |z_1 - z_3|$ , then*

$$\left| \frac{w_1 - w_3}{w_1 - w_2} \right|^\varepsilon \prec \left| \frac{z_1 - z_3}{z_1 - z_2} \right| \prec \left| \frac{w_1 - w_3}{w_1 - w_2} \right|^c$$

where  $0 < r_0 < 1$  a constant, depending on  $G$  and  $k$ .

**Lemma 11** *Let  $G$  be a  $k$ -quasidisk for some  $0 \leq k < 1$ . Then*

$$|\Psi(w_1) - \Psi(w_2)| \succ |w_1 - w_2|^{1+k},$$

for all  $w_1, w_2 \in \overline{\Omega}'$ .

This fact it follows from of an appropriate result for the mapping  $f \in \Sigma(k)$  [9, p.287] and estimation for the  $\Psi'$  [5, Th.2.8]

**Lemma 12** *Let  $G$  be a quasidisk and  $P_n(z)$ ,  $\deg P_n \leq n$ ,  $n = 1, 2, \dots$ , is arbitrary polynomial and weight function  $h(z)$  satisfied the condition (5). Then for any  $R > 1$ ,  $p > 0$  and  $n = 1, 2, \dots$*

$$\|P_n\|_{A_p(h, G_{1+c(R-1)})} \leq c_1 R^{n+\frac{2}{p}} \|P_n\|_{A_p(h, G)} \quad (11)$$

where  $c, c_1$  are independent of  $n$  and  $G$ .

This lemma in case of  $p = 2$  mentioned in [1]. In particular, in  $h(z) = 1$  we get

$$\|P_n\|_{A_p(G_{1+c(R-1)})} \leq c_1 R^{n+\frac{2}{p}} \|P_n\|_{A_p(G)} \quad (12)$$

This result is the integral analog of the familiar lemma of Bernstein-Walsh [13, p.101] for the case  $A_p(G)$ -norm and, shows that the order  $A_p(G)$ -norm of arbitrary polynomials is taken from the region  $G$  and  $G_{1+1/n}$  which both have the identical order. For the case of  $L_p(\partial G)$ -norm the appropriate result has been proven in [8].

## 4 The Proof of Theorems.

Let  $P_n(z)$  is arbitrary polynomials of degree at most  $n$ ;  $M_{n,p} := \|P_n\|_{A_p(h, G)}$ . For the sake of simplicity, we assume that the  $m = 1$ ,  $\gamma_1 \equiv \gamma$ ,  $\nu_1 \equiv \nu$ .

For each  $R > 1$  let  $L^* := y(L_R)$ ,  $G^* := \text{int} L^*$ ,  $\Omega^* := \text{ext} L^*$ .

According to [4], for all  $z \in L^*$  and  $t \in L$  such that  $|z - t| = d(z, L_R)$  we have

$$d(z, L) \asymp d(t, L_R) \asymp d(z, L_R) \quad (13)$$

Since  $L$  is a quasicircle, then any level curve  $L_R$ ,  $R = 1 + cn^{-1}$  also is quasicircle. Therefore, there exist a  $K_1$ -quasiconformal reflection  $y_R(\zeta)$ ,  $y_R(0)$

$= \infty$  across  $L_R$  such that it satisfies the condition(10) described by  $y_R(\zeta)$ . For  $y_R(\zeta)$ , we can write for  $P_n(z)$  the following integral representation[5, p.105]

$$P_n(z) = -\frac{1}{\pi} \iint_{G_R} \frac{P_n(\zeta) y_{R,\bar{\zeta}}(\zeta)}{(y_R(\zeta) - z)^2} d\sigma_\zeta, z \in G_R. \quad (14)$$

For  $\varepsilon > 0$  by setting  $U_\varepsilon(z) := \{\zeta : |\zeta - z| < \varepsilon\}$  and without loss of generalite we may take  $U_\varepsilon := U_\varepsilon(0) \subset G^*$ .

For  $z \in L$  we have

$$|P_n(z)| \leq \frac{1}{\pi} \left\{ \iint_{U_\varepsilon} + \iint_{G_R \setminus U_\varepsilon} \right\} \frac{|P_n(\zeta)| |y_{R,\bar{\zeta}}(\zeta)|}{|y_R(\zeta) - z|^2} d\sigma_\zeta =: J_1 + J_2 \quad (15)$$

To estimate the integral  $J_1$ , we multiply the numerator and denominator of integrant to  $h^{1/p}(\zeta)$ , and applying the Holder inequality we get

$$J_1 \prec \left\{ \iint_{U_\varepsilon} h(\zeta) |P_n(\zeta)|^p d\sigma_\zeta \right\}^{\frac{1}{p}} \left\{ \iint_{U_\varepsilon} \frac{|y_{R,\bar{\zeta}}(\zeta)|^q}{h^{q-1}(\zeta) |y_R(\zeta) - z|^{2q}} d\sigma_\zeta \right\}, \frac{1}{p} + \frac{1}{q} = 1.$$

The first multiplier smaller than  $\pi^{-1} M_{n,p}$ . According to (10)  $|y_{R,\bar{\zeta}}| \asymp |y_R(\zeta)|^2$ , for all  $\zeta \in U_\varepsilon$ , because of  $|\zeta - z_1| \geq \varepsilon$ ,  $|y_R(\zeta) - z| \asymp |y_R(\zeta)|$  for  $z \in L$  and  $\zeta \in U_\varepsilon$ , then we can find

$$J_1 \prec M_{n,p} \quad (16)$$

For the estimation of  $J_2$ , first of all we note that the Jacobian  $\mathcal{L}_{y_R} := |y_{R,\zeta}|^2 - |y_{R,\bar{\zeta}}|^2$  of the reflection  $y_R(\zeta)$  satisfied the following inequality

$$\begin{aligned} |y_{R,\bar{\zeta}}| &= \left[ \frac{\mathcal{L}_{y_R} |y_{R,\zeta}|^2}{|y_{R,\zeta}|^2 - |y_{R,\bar{\zeta}}|^2} \right]^{\frac{1}{2}} = \left[ \frac{\mathcal{L}_{y_R}}{\left( |y_{R,\zeta}|^2 / |y_{R,\bar{\zeta}}|^2 \right) - 1} \right]^{\frac{1}{2}} \\ &\leq \left( \frac{\chi^2}{1 - \chi^2} \right)^{\frac{1}{2}} |\mathcal{L}_{y_R}|^{\frac{1}{2}} \prec |\mathcal{L}_{y_R}|^{\frac{1}{2}}, \end{aligned} \quad (17)$$

where  $\chi := \frac{K_1-1}{K_1+1}$ . Next, analogously to the estimate for  $J_1$ , after the carrying out that the change of variable we obtained

$$\begin{aligned} J_2 &\prec M_{n,p} \left\{ \iint_{G_R \setminus U_\varepsilon} \frac{d\sigma_\zeta}{|\zeta - z_1|^{\gamma(q-1)} |y_R(\zeta) - z|^{2q}} \right\}^{\frac{1}{q}} \\ &\prec M_{n,p} \left\{ \iint_{y_R(G_R \setminus U_\varepsilon)} \frac{d\sigma_\zeta}{|y_R(\zeta) - z_1|^{\gamma(q-1)} |\zeta - z|^{2q}} \right\}^{\frac{1}{q}}, \end{aligned} \quad (18)$$

from (10),(17) and Lemma12.

Let  $\gamma = 0$ , i.e the boundary  $L$  and weight function  $h(z)$  do not possess singularity. In this case in (18) we get

$$J_2 \prec M_{n,p} d^{-\frac{2}{p}}(z, L_R) \quad (19)$$

From (15),(16) and (19) we obtained

$$|P_n(z)| \prec M_{n,p} d^{-\frac{2}{p}}(z, L_R) \quad (20)$$

Since  $M_{n,2} \equiv 1$  for the  $K_n(z)$ , then using Lemma11 we get the prove of Theorem2, Theorem4 and the first part of Theorem9.

Let now  $\gamma \neq 0$ . First of all we shall establish that

$$|\zeta - z_1| \prec |y_R(\zeta) - z_1| \quad (21)$$

for all  $\zeta \in G_R \setminus U_\varepsilon$  and  $z_1 \in L$ .

In fact, let  $|z_1 - t| = d(z_1, L_R)$ ,  $t \in L_R$ . According by (10)

$$c_1 |\zeta - z| \leq |y_R(\zeta) - z| \leq c_2 |\zeta - z|, \quad (22)$$

for all  $\zeta \in G_R \setminus U_\varepsilon$  and  $z \in L_R$ , then

$$\begin{aligned} |\zeta - z_1| &\leq |\zeta - t| + |y_R(\zeta) - t| + |y_R(\zeta) - z_1| \\ &\leq (c_1^{-1} + 1) |y_R(\zeta) - t| + |y_R(\zeta) - z_1| \\ &\prec |y_R(\zeta) - z_1|. \end{aligned}$$

If  $\gamma < 0$ , according to (21), after the carrying out that the change of variable  $\zeta = y_R(\zeta)$ , and using (17) and (10), from (18) we have

$$\begin{aligned} J_2 &\prec M_{n,p} \left\{ \iint_{G_R \setminus U_\varepsilon} \frac{d\sigma_\zeta}{|y_R(\zeta) - z_1|^{\gamma(q-1)} |y_R(\zeta) - z|^{2q}} \right\}^{\frac{1}{q}} \\ &\prec M_{n,p} \left\{ \iint_{y_R(G_R \setminus U_\varepsilon)} \frac{d\sigma_\zeta}{|\zeta - z_1|^{\gamma(q-1)} |\zeta - z|^{2q}} \right\}^{\frac{1}{q}}. \end{aligned}$$

If  $\gamma > 0$ , by changing the variable  $\zeta = y_R(\zeta)$  and applying (21), (17) and (10) we obtained

$$J_2 \prec M_{n,p} \left\{ \iint_{y_R(G_R \setminus U_\varepsilon)} \frac{d\sigma_\zeta}{|\zeta - z_1|^{\gamma(q-1)} |\zeta - z|^{2q}} \right\}^{\frac{1}{q}} \quad (23)$$

We denote with  $\widetilde{J}_2$  which the integral in braces and we estimate  $\widetilde{J}_2$  in different situations of the points  $z$  and  $z_1$  on  $L$ . For this we set

$$\begin{aligned} E_0 &:= E_1 \cup E_2; \\ E_1 &:= \{y_R(G_R \setminus U_\varepsilon) \cap U_\delta(z_1)\}, E_2 := \{y_R(G_R \setminus U_\varepsilon) \setminus U_\delta(z_1)\}, 0 < \delta < \delta_0(G); \\ E_{i1} &:= \{\zeta \in E_i : |\zeta - z_1| \geq |\zeta - z|\}, E_{i2} := \{\zeta \in E_i : |\zeta - z_1| < |\zeta - z|\}, \\ i &= 0, 1, 2; \\ E'_i &:= \Phi(E_i), w = \Phi(z), w_1 = \Phi(z_1), \tau = \Phi(\zeta). \end{aligned}$$

Let  $|z - z_1| < \delta$ . In integral  $\widetilde{J}_2$ , we change the variable  $\tau = \Phi(\zeta)$ , and according to 10 we have

$$\begin{aligned} \widetilde{J}_2 &= \iint_{\Phi(y_R(G_R \setminus U_\varepsilon))} \frac{|\Psi'(\tau)| d\sigma_\tau}{|\Psi(\tau) - z_1|^{\gamma(q-1)} |\Psi(\tau) - z|^{2q}} \\ &\asymp \iint_{\Phi(y_R(G_R \setminus U_\varepsilon))} \frac{d^2(\Psi(\tau), L) d\sigma_\tau}{|\Psi(\tau) - z_1|^{\gamma(q-1)} |\Psi(\tau) - z|^{2q} (|\tau| - 1)^2} \\ &= \left( \iint_{E_1} + \iint_{E_2} \right) \frac{d^2(\Psi(\tau), L) d\sigma_\tau}{|\Psi(\tau) - \Psi(w_1)|^{\gamma(q-1)} |\Psi(\tau) - \Psi(w)|^{2q} (|\tau| - 1)^2} \\ &=: \widetilde{J}_{21} + \widetilde{J}_{22} \end{aligned} \tag{24}$$

from [5, Th.2.8]

We estimate the last integrals separately:

$$\begin{aligned} \widetilde{J}_{21} &\leq \iint_{E'_{11}} \frac{|\Psi(\tau) - \Psi(w)|^2 d\sigma_\tau}{|\Psi(\tau) - \Psi(w)|^{2q+\gamma(q-1)} (|\tau| - 1)^2} \\ &\quad + \iint_{E'_{12}} \frac{|\Psi(\tau) - \Psi(w_1)|^2 d\sigma_\tau}{|\Psi(\tau) - \Psi(w_1)|^{2q+\gamma(q-1)} (|\tau| - 1)^2} \\ &\prec \iint_{E'_{11}} \frac{d\sigma_\tau}{|\tau - w|^{\frac{(2+\gamma)(q-1)}{\beta} - \frac{2q}{\alpha p}} |\tau - w|^{\frac{2q}{\alpha p}} (|\tau| - 1)^2} \\ &\quad + \iint_{E'_{12}} \frac{d\sigma_\tau}{|\tau - w_1|^{\frac{(2+\gamma)(q-1)}{\beta} - \frac{2q}{\alpha p}} |\tau - w_1|^{\frac{2q}{\alpha p}} (|\tau| - 1)^2} \end{aligned}$$

According by (6) we get

$$\widetilde{J}_{21} \prec \iint_{E'_{11}} \frac{d\sigma_\tau}{(|\tau| - 1)^{2+\frac{2q}{\alpha p}}} + \iint_{E'_{12}} \frac{d\sigma_\tau}{(|\tau| - 1)^{2+\frac{2q}{\alpha p}}} \prec n^{\frac{2q}{\alpha p}}. \tag{25}$$

For estimation of  $\widetilde{J}_{22}$  we note that  $|\zeta - z_1| \geq \delta$  and therefore, according Lemma10 we have

$$\widetilde{J}_{22} \prec \iint_{E'_2} \frac{|\Psi(\tau) - \Psi(w)|^2 d\sigma_\tau}{|\Psi(\tau) - \Psi(w)|^{2q} (|\tau| - 1)^2} \prec \iint_{E'_2} \frac{d\sigma_\tau}{(|\tau| - 1)^{2(q-1)+2}} \prec n^{\frac{2(q-1)}{\alpha}}. \quad (26)$$

Let  $|z - z_1| \geq \delta$ . In this case we see that

$$\begin{aligned} |\zeta - z| &\geq \delta/2 \text{ for } \zeta \in E_1; |\zeta - z_1| \geq \delta/2 \text{ for } \zeta \in E_{21}, \\ |\zeta - z| &\geq \delta/2 \text{ and } |\zeta - z_1| \geq \delta \text{ for } \zeta \in E_{22} \end{aligned}$$

then for  $\widetilde{J}_2$  we obtained

$$\begin{aligned} \widetilde{J}_2 &\prec \iint_{E_1} \frac{d\sigma_\zeta}{|\zeta - z_1|^{\gamma(q-1)}} + \iint_{E_{21}} \frac{d\sigma_\zeta}{|\zeta - z|^{2q}} \\ &\prec n^{\frac{\gamma(q-1)-2}{\beta}} + n^{\frac{2(q-1)}{\alpha}} + 1 \prec n^{\frac{2(q-1)}{\alpha}}. \end{aligned} \quad (27)$$

Substituting (27), (26) and (25), in (??), and after the obtained estimation for  $\widetilde{J}_2$  writing into (23) we get

$$J_2 \prec M_{n,p} n^{\frac{2}{\alpha p}}. \quad (28)$$

According (28) and (16) in (15), we complete the proof of Theorem9 and, consequently, the proof of Theorem7.

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# The Properties of the Orthogonal Polynomials with Weight Having Singularity on the Boundary Contour

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## Abstract

In this work the order of the height of orthogonal polynomials over the region with respect to the weight is analyzed, when the boundary contour and the weight functions have some singularities.

*Keywords:* Orthogonal polynomials, Weight function, Quasicycle, Conformal mapping, Quasiconformal mapping.

*Subject Classes:* 30A10, 30C10 41A17.

## 1 Introduction and Definitions.

Let  $G$  be a finite region, with  $0 \in G$ , bounded by a Jordan curve  $L := \partial G$  and  $h(z)$  is a weight function in  $G$ . The polynomials  $\{K_n(z)\}$ ,  $\deg K_n = n$ ,  $n = 0, 1, 2, \dots$ , satisfying the condition

$$\iint_G h(z) K_n(z) \overline{K_m(z)} d\sigma_z = \delta_{n,m}$$

are called orthonormal polynomials for the pair  $(G, h)$ . They are determined uniquely if the coefficient of the term of the highest degree is positive.

Let  $\{z_i\}$ ,  $i = \overline{1, m}$  is the fixed system of the points on  $L$  and the weight function  $h(z)$  defined as the following:

$$h(z) = h_0(z) \prod_{i=1}^m |z - z_i|^{\gamma_i}, \quad (1)$$

where  $\gamma_i > -2$  and  $h_0(z)$  is satisfying the condition  $h_0(z) \geq c > 0$ .

In this work we study the order of the height of polynomials  $K_n(z)$  on boundary points of the region, where the boundary contour  $L$  and the weight function  $h(z)$  has some singularities. The similiar problems have been studied in [10],[12],[13] in case of orthogonality along a curve and in [1]-[5],[14] in case of orthogonality over the region.

Let's give some definitions.

Throughout this paper  $c, c_1, c_2, \dots$  are positive, and  $\varepsilon, \varepsilon_1, \varepsilon_2, \dots$  sufficiently small positive constants (in general, different in different relations), which depend on  $G$  in general.

For  $\delta > 0$  and  $z \in C$  let us set :  $B(z, \delta) := \{\zeta : |\zeta - z| < \delta\}$ ,  $B := B(0, 1)$ ,  $\Delta(z, \delta) := \text{ext } \overline{B(z, \delta)} = \{\zeta : |\zeta - z| > \delta\}$ ,  $\Delta := \text{ext } B$ ,  $\Omega := \text{ext } G$ ,  $\Omega(z, \delta) := \Omega \cap B(z, \delta)$ ;  $w = \varphi(z)$  ( $w = \Phi(z)$ ) be the conformal mapping of  $G$  ( $\Omega$ ) onto the  $B(\Delta)$  normalized by  $\varphi(0) = 0, \varphi'(0) > 0$  ( $\Phi(\infty) = \infty, \Phi'(\infty) > 0$ ),  $\psi := \varphi^{-1}$  ( $\Psi := \Phi^{-1}$ ).

**Definition 1** A bounded Jordan region  $G$  is called a  $k$ -quasidisk,  $0 \leq k < 1$ , if any conformal mapping  $\psi$  can be extended to a  $K$ -quasiconformal,  $K = \frac{1+k}{1-k}$ , homeomorphism of the plane  $\overline{C}$  on the  $\overline{C}$ . In that case the curve  $L := \partial G$  is called a  $K$ -quasicircle. The region  $G$  (curve  $L$ ) is called a quasidisk (quasicircle), if it is  $k$ -quasidisk ( $k$ -quasicircle) with some  $0 \leq k < 1$ .

**Definition 2** We say that  $G \in Q_\alpha$ ,  $0 < \alpha \leq 1$ , if

- a)  $L$  is a quasicircle,
- b)  $\Phi \in \text{Lip } \alpha$ ,  $z \in \overline{\Omega}$ .

**Definition 3** We say that  $\Omega \in Q(\nu)$ ,  $0 < \nu < 1$ , if

- i)  $L := \partial\Omega = \partial G$  is quasicircle,
- ii) For  $\forall z \in L$ , there exists a  $r > 0$  and  $0 < \nu < 1$  such that a closed circular sector  $S(z; r, \nu) := \{\zeta : \zeta = z + re^{i\theta}, 0 \leq \theta_0 < \theta < \theta_0 + \nu\}$  of radius  $r$  and opening  $\nu\pi$  lies in  $\overline{G}$  with vertex at  $z$ .

**Definition 4** We say that  $\Omega \in Q_\alpha(\nu_1, \dots, \nu_m)$ ,  $0 < \nu_1, \dots, \nu_m < \alpha \leq 1$ , if there exist a system of points  $\{z_i\}$ ,  $i = \overline{1, m}$  on  $L$ , such that  $\Omega \in Q(\nu_i)$  for any points  $z_i \in L$  and  $\Phi \in \text{Lip } \alpha$ ,  $z \in \overline{\Omega} \setminus \{z_i\}$ .

Assume that the system of points  $\{z_i\}$ ,  $i = \overline{1, m}$  mentioned in (1) and Definition 4 is identically ordered on  $L$ . In [2] we showed, that if interference conditions

$$1 + \frac{\gamma_i}{2} = \frac{1}{\alpha(2 - \nu_i)} \quad (2)$$

is satisfied for any singular points  $\{z_i\}$ ,  $i = \overline{1, m}$  of the weight functions and boundary contour, then the order of the height of polynomials  $K_n(z)$  in  $\overline{G}$  acts itself identically neither weight  $h(z)$  and boundary contour  $L$  have not got singularity nor they have got singularity.

In the present paper we investigate the case when (2) is not satisfied.

## 2 Main results.

**Theorem 5** *Let  $\Omega \in Q_\alpha(\nu_1)$ , for some  $0 < \nu_1 < 1$  and  $\alpha(2 - \nu_1) \geq 1$ ;  $h(z)$  defined by (1). If*

$$1 + \frac{\gamma_1}{2} < \frac{1}{\alpha(2 - \nu_1)}, \quad (3)$$

*then for every  $z \in \overline{G}$  and each  $n = 1, 2, \dots$*

$$|K_n(z)| \leq c_1 n^{s_1} + c_2 |z - z_1|^{\sigma_1} n^{1/\alpha}, \quad (4)$$

*where*

$$s_1 = \frac{(2 + \gamma_1)(2 - \nu_1)}{2}, \quad \sigma_1 = \frac{1}{\alpha(2 - \nu_1)} - \frac{2 + \gamma_1}{2}. \quad (5)$$

Since  $\alpha(2 - \nu_1) \geq 1$ , (3) will be satisfied when  $-2 < \gamma_1 < 0$ . Here and from (4) we see that the order of the height of  $K_n$  in point  $z_1$  and points  $z \in L$ ,  $z \neq z_1$  where  $h(z) \rightarrow \infty$  and curve  $L$  doesn't have singularity, acts itself identically. Thus, the conditions (3) we will call algebraic pole conditions of the order  $\lambda_1 = 1 - \alpha(2 - \nu_1)(1 + \frac{\gamma_1}{2})$ .

This theorem can be extended to case when  $L$  and  $h(z)$  have a lot of singular points. For example, in case of two singular points we can write

$$|K_n(z)| \leq c_1 |z - z_1|^{\sigma_1} n^{s_2} + c_2 |z - z_2|^{\sigma_2} n^{s_1} + c_3 |z - z_1|^{\sigma_1} |z - z_2|^{\sigma_2} n^{1/\alpha}, \quad z \in \overline{G}, \quad (6)$$

where  $s_i, \sigma_i$ ,  $i = 1, 2$ , defined as it is in (5) respectively.

Theorem 5 also is correct if the curve  $L$  has at point  $z_1$  algebraic pole and at points  $\{z_k\}$ ,  $k \geq 2$ , singularities, which in satisfying the interference conditions (2).

**Theorem 6** Let  $\Omega \in Q_\alpha(\nu_1, \dots, \nu_m)$ , for some  $0 < \nu_i < 1$  and  $\alpha(2 - \nu_i) \geq 1$ ,  $h(z)$  defined by (1). If

$$1 + \frac{\gamma_i}{2} > \frac{1}{\alpha(2 - \nu_i)} \quad (7)$$

is satisfied for any points  $z_i \in L$ ,  $i = \overline{1, m}$ . Then, for every  $n = 1, 2, \dots$

$$\max_{z \in \overline{G}} \left( \prod_{i=1}^m |z - z_i|^{\tilde{\mu}_i} |K_n(z)| \right) \leq c_4 n^{1/\alpha}, \quad (8)$$

$$|K_n(z_i)| \leq c_5 n^{\tilde{s}_i}, \quad (9)$$

where

$$\tilde{\mu}_i := 1 + \frac{\gamma_i}{2} - \frac{1}{\alpha(2 - \nu_i)} \quad (10)$$

$$\tilde{s}_i := \left(1 + \frac{\gamma_i}{2}\right)(2 - \nu_i), \quad i = \overline{1, m}. \quad (11)$$

The conditions (7) will be satisfied sometimes when  $\gamma_i > 0$ ,  $i = \overline{1, m}$ . Thus, (7) we will call algebraic zero conditions of the order  $\mu_i = \alpha(2 - \nu_i) \left(1 + \frac{\gamma_i}{2}\right) - 1$ .

### 3 Some auxiliary results

In the following we shall use the notations “ $a \prec b$ ” and “ $a \asymp b$ ” are equivalent to  $a \leq b$  and  $c_1 a \leq b \leq c_2 a$  for some constants  $c, c_1, c_2$  respectively.

Let  $G$  is a quasidisk. Then there exists a quasiconformal reflection  $y(\cdot)$  across  $L$  such that  $y(G) = \Omega$ ,  $y(\Omega) = G$  and  $y(\cdot)$  fixes the points of  $L$ . The quasiconformal reflection  $y(\cdot)$  is such that it satisfied the following condition [7, p.26]

$$\begin{aligned} |y(\zeta) - z| &\asymp |\zeta - z|, \quad z \in L, \quad \varepsilon < |\zeta| < \frac{1}{\varepsilon} \\ |y_{\bar{\zeta}}| &\asymp |y_{\zeta}| \asymp 1, \quad \varepsilon < |\zeta| < \frac{1}{\varepsilon} \\ |y_{\bar{\zeta}}| &\asymp |y(\zeta)|^2, \quad |\zeta| < \varepsilon, \quad |y_{\bar{\zeta}}| \asymp |\zeta|^{-2}, \quad |\zeta| > \frac{1}{\varepsilon}. \end{aligned} \quad (12)$$

For  $t > 0$ , let  $L_t := \{z : |\varphi(z)| = t, \text{ if } t < 1, |\Phi(z)| = t, \text{ if } t > 1\}$ ,  $G_t := \text{int} L_t$ ,  $\Omega_t := \text{ext} L_t$  and for  $t > 1$  let  $L^* := y(L_t)$ ,  $G^* := \text{int} L^*$ ,  $\Omega^* :=$

$extL^*$ ;  $w = \Phi_R(z)$  be the conformal mapping of  $\Omega^*$  onto the  $\Delta$  normalized by  $\Phi_R(\infty) = \infty, \Phi_R'(\infty) > 0$ ;  $\Psi_R := \Phi_R^{-1}$ ;  $L_t^* := \{z : |\Phi_R(z)| = t\}$ ,  $G_t^* := intL_t^*$ ,  $\Omega_t^* := extL_t^*$ ;  $d(z, L) := dist(z, L)$ .

According to [6], for all  $z \in L^*$  and  $t \in L$  such that  $|z - t| = d(z, L_R)$  we have

$$d(z, L) \asymp d(t, L_R) \asymp d(z, L_R) \quad (13)$$

**Lemma 7** [5] *Let  $G$  be a quasidisk,  $z_1 \in L$ ,  $z_2, z_3 \in \Omega \cap \{z : |z - z_1| \prec d(z_1, L_{r_0})\}$ ;  $w_j = \Phi(z_j)$ ,  $j = 1, 2, 3$ . Then*

a) *The statements  $|z_1 - z_2| \prec |z_1 - z_3|$  and  $|w_1 - w_2| \prec |w_1 - w_3|$  are equivalent. So are  $|z_1 - z_2| \asymp |z_1 - z_3|$  and  $|w_1 - w_2| \asymp |w_1 - w_3|$ .*

b) *If  $|z_1 - z_2| \prec |z_1 - z_3|$ , then*

$$\left| \frac{w_1 - w_3}{w_1 - w_2} \right|^\varepsilon \prec \left| \frac{z_1 - z_3}{z_1 - z_2} \right| \prec \left| \frac{w_1 - w_3}{w_1 - w_2} \right|^c$$

where  $0 < r_0 < 1$  a constant, depending on  $G$  and  $k$ .

Let  $A_p(h, G)$ ,  $p > 0$  denotes the class of function  $f$  which analytic in  $G$  and satisfying the condition

$$\|f\|_{A_p} := \|f\|_{A_p(h, G)} := \left( \iint_G h(z) |f(z)|^p d\sigma_z \right)^{1/p} < \infty.$$

**Lemma 8** *Let  $p > 0$ ;  $f$  be an analytic function in  $|z| > 1$  and has at  $z = \infty$  pole of degree at most  $n$ ,  $n \geq 1$ . Then for all  $R_1, R_2, 1 < R_1 < R_2$*

$$\|f\|_{A_p(R_1 < |z| < R_2)} \leq \left( 2 \frac{R_2 - R_1}{R_1 - 1} \right)^{1/p} R_2^{n + \frac{2}{p}} \|f\|_{A_p(1 < |z| < R_1)}$$

**Proof.** According to Riesz theorem [9, p.443], for any  $\rho$ ,  $R_1 \leq \rho < R_2$  and  $s$ ,  $1 < s \leq R_1$  we can write

$$\int_{|z|=\rho} \left| \frac{f(z)}{z^{n+\frac{1}{p}}} \right|^p |dz| \leq \int_{|z|=R_1} \left| \frac{f(z)}{z^{n+\frac{1}{p}}} \right|^p |dz| \quad (14)$$

$$\int_{|z|=R_1} \left| \frac{f(z)}{z^{n+\frac{1}{p}}} \right|^p |dz| \leq \int_{|z|=s} \left| \frac{f(z)}{z^{n+\frac{1}{p}}} \right|^p |dz| \quad (15)$$

respectively. Integrate (14) over the  $\rho$  from  $R_1$  to  $R_2$ , and (15) over the  $s$  from 1 to  $R_1$  we get

$$\iint_{R_1 < |z| < R_2} |f(z)|^p d\sigma_z \leq \frac{R_2^{np+2} - R_1^{np+2}}{R_1^{np+2} - 1} \iint_{1 < |z| < R_1} |f(z)|^p d\sigma_z \quad (16)$$

Let  $np =: m \in N \cup \{0\}$ . Then, setting  $R_3 := R_2/R_1$  we obtain

$$S := \frac{R_2^{np+2} - R_1^{np+2}}{R_1^{np+2} - 1} \leq R_1^{m+2} \frac{(R_3 - 1)(R_3^{m+1} + \dots + 1)}{(R_1 - 1)(R_1^{m+1} + \dots + 1)} \leq \frac{R_2 - R_1}{R_1 - 1} R_2^{m+1}. \quad (17)$$

Let  $m \notin N$ . Then according the trivial equality  $k := [m] \leq m < [m] + 1$  in case of  $m \geq 1$  we have

$$\begin{aligned} S &\leq R_1^{k+3} \frac{(R_3^{k+3} - 1)}{(R_1^{k+2} - 1)} \leq R_1^{k+3} \frac{(R_3 - 1)(R_3^{k+2} + \dots + 1)}{(R_1 - 1)(R_1^{k+1} + \dots + 1)} \\ &\leq \frac{R_2 - R_1}{R_1 - 1} \frac{k+3}{k+2} R_2^{k+2} \leq 2 \frac{R_2 - R_1}{R_1 - 1} R_2^{m+2}, \end{aligned} \quad (18)$$

and in case of  $m < 1$  we get

$$S \leq \frac{(R_3^3 - 1)}{(R_1^2 - 1)} R_1^3 \leq \frac{R_3 - 1}{R_1 - 1} \frac{3R_3^2}{2} R_1^3 < 2 \frac{R_2 - R_1}{R_1 - 1} R_2^2 \quad (19)$$

According (17)-(19) in (16) we complete the proof.

**Lemma 9** *Let  $G$  be a quasidisk and  $P_n(z)$ ,  $\deg P_n \leq n$ ,  $n = 1, 2, \dots$ , is arbitrary polynomial and weight function  $h(z)$  satisfied the condition(1). Then for any  $R > 1$ ,  $p > 0$  and  $n = 1, 2, \dots$*

$$\|P_n\|_{A_p(h, G_{1+c(R-1)})} \leq c_1 R^{n+\frac{2}{p}} \|P_n\|_{A_p(h, G)} \quad (20)$$

where  $c, c_1$  are independent of  $n$  and  $R$ .

**Proof.** We present the proof of (20) under several headings. First of all, it is easy to convince ourselves that for the proof of (20) sufficiently show the fulfilment of the following estimations

$$\|P_n\|_{A_p(h, G_R \setminus G)} \prec [1 + c(R-1)]^{n+\frac{2}{p}} \|P_n\|_{A_p(h, G \setminus G^*)} \quad (21)$$



for some  $c > 0$ .

Now, we consider the two numbers  $\rho_1, \rho_2$ ,  $\rho_1 < \rho_2$  such that

$$G_{\rho_1}^* \subset G \quad (22)$$

$$G_R \subset G_{\rho_2}^*, \quad (23)$$

and show that we can choose the numbers  $\rho_1, \rho_2$  that they satisfy the following conditions

$$\rho_1 - 1 \asymp R - 1; \quad (24)$$

$$\rho_2 - 1 \asymp R - 1. \quad (25)$$

In fact, let us  $\rho_1, \rho_2$  are arbitrary numbers satisfying (22) and (23),  $z \in L^*$ ,  $\tilde{z} = y(z)$ . The points  $z_1 \in L_{\rho_1}^*$ ,  $z_2 \in L$  and  $z_3 \in L_{\rho_2}^*$  we define as  $d(z, L_{\rho_1}^*) = |z - z_1|$ ;  $d(z, L) = |z - z_2|$  and  $d(z, L_{\rho_2}^*) = |z - z_3|$  respectively. According to (13), there exist  $c_3, c_4$  that are independent from  $z$  and  $R$  such that

$$c_3 d(z_2, L_R) \leq d(z, L) \leq c_4 d(z_2, L_R) \quad (26)$$

Since  $L^*$  is a quasicircle, applying Lemma 7 to functions  $\Phi_R$  we obtain

$$\left| \frac{z - z_2}{z - z_1} \right| \geq c_5 \left| \frac{\Phi_R(z) - \Phi_R(z_2)}{\Phi_R(z) - \Phi_R(z_1)} \right|^{\varepsilon_1} \geq c_6 \left( \frac{|\Phi_R(z) - \Phi_R(z_2)|}{\rho_1 - 1} \right)^{\varepsilon_1},$$

and, from this we get

$$|z - z_1| \leq c_6^{-1} \left( \frac{\rho_1 - 1}{|\Phi_R(z) - \Phi_R(z_2)|} \right)^{\varepsilon_1} |z - z_2|. \quad (27)$$

Using the D-property of the mapping  $y_R(z)$  [8, p.18] we have

$$|z - z_2| \geq c_3 d(z_2, L_R) \geq c_7 |\tilde{z} - z_2|.$$

and, according Lemma 7 we get

$$|\Phi_R(z) - \Phi_R(z_2)| \geq c_8 |\Phi_R(\tilde{z}) - \Phi_R(z_2)| \geq c_8 (R - 1)$$

Then, from (27) we obtain

$$|z - z_1| \leq c_6^{-1} \left( \frac{\rho_1 - 1}{c_8 (R - 1)} \right)^{\varepsilon_1} |z - z_2|.$$

So, we can take

$$\rho_1 = 1 + c_9(R - 1) \quad (28)$$

with  $c_9 = \frac{1}{2}c_8 \cdot c_6^{-\varepsilon_1}$ , which also leads to (22) and (24).

We now define  $\rho_2$ . For this applying Lemma 7 to  $\Phi_R$  we get

$$\left| \frac{z - \tilde{z}}{z - z_3} \right| \leq c_{10} \left| \frac{\Phi_R(z) - \Phi_R(\tilde{z})}{\Phi_R(z) - \Phi_R(z_3)} \right|^c$$

and from this we obtain

$$|z - z_3| \geq c_{11} \left( \frac{\rho_2 - 1}{|\Phi_R(z) - \Phi_R(\tilde{z})|} \right)^c |z - \tilde{z}|. \quad (29)$$

Since  $|\Phi_R(z) - \Phi_R(z_2)| \leq c_{12} |\Phi_R(\tilde{z}) - \Phi_R(z_2)|$ , then

$$\begin{aligned} |\Phi_R(z) - \Phi_R(\tilde{z})| &\leq |\Phi_R(z) - \Phi_R(z_2)| + |\Phi_R(\tilde{z}) - \Phi_R(z_2)| \\ &\leq (c_{12} + 1) |\Phi_R(\tilde{z}) - \Phi_R(z_2)| \leq c_{13}(R - 1), \end{aligned}$$

and from (29) we have

$$|z - z_3| \geq c_{11} \left( \frac{\rho_2 - 1}{c_{13}(R - 1)} \right)^c |z - \tilde{z}|.$$

Choosing

$$\rho_2 = 1 + c_{14}(R - 1) \quad (30)$$

with  $c_{14} = c_8 \cdot c_6^{-\varepsilon_1} + c_{13} \cdot c_{11}^{c-1}$ , we see that the (23) and (25) are satisfied.

Now, let's make a proof of (21). Let's include the functions Blaschke with respect to the singular points of the weight functions  $h(z)$

$$B_R(z) = \prod_{i=1}^m B_R^i(z) := \prod_{i=1}^m \frac{\Phi_R(z) - \Phi_R(z_i)}{1 - \overline{\Phi_R(z_i)} \Phi_R(z)}, \quad z \in \Omega^*. \quad (31)$$

It is easy that the  $B_R(z_i) = 0$  and  $|B_R(z)| \equiv 1$  at  $z \in L^*$ .

For the  $p > 0$  and  $R > 1$  let us set

$$f_R(w) := h_0(\Psi_R(w)) \prod_{i=1}^m \left[ \frac{\Psi_R(w) - \Psi_R(w_i)}{w B_R^i(\Psi_R(w))} \right]^{\frac{\gamma_i}{p}} P_n(\Psi_R(w)) \left[ \Psi_R'(w) \right]^{\frac{2}{p}}, \quad w = \Phi_R(z).$$

The function  $f_R$  is analytic in  $\Delta$  and have pole of degree at most  $n$  on  $z = \infty$ . Then, according to Lemma 8 we have

$$\|f_R\|_{A_p(\rho_1 < |w| < \rho_2)} \leq \left(2 \frac{\rho_2 - \rho_1}{\rho_1 - 1}\right)^{1/p} \rho_2^{n+\frac{2}{p}} \|f_R\|_{A_p(1 < |w| < \rho_1)},$$

or

$$\begin{aligned} & \iint_{G_R \setminus G} h_0(z) \prod_{i=1}^m \left| \frac{z - z_i}{\Phi_R(z) B_R^i(z)} \right|^{\gamma_i} |P_n(z)|^p d\sigma_z \\ & \leq \iint_{G_{\rho_2}^* \setminus G_{\rho_1}^*} h_0(z) \prod_{i=1}^m \left| \frac{z - z_i}{\Phi_R(z) B_R^i(z)} \right|^{\gamma_i} |P_n(z)|^p d\sigma_z \\ & \leq 2 \frac{\rho_2 - \rho_1}{\rho_1 - 1} \rho_2^{pn+2} \iint_{G_{\rho_1}^* \setminus G^*} h_0(z) \prod_{i=1}^m \left| \frac{z - z_i}{\Phi_R(z) B_R^i(z)} \right|^{\gamma_i} |P_n(z)|^p d\sigma_z \\ & \leq 2 \frac{\rho_2 - \rho_1}{\rho_1 - 1} \rho_2^{pn+2} \iint_{G \setminus G^*} h_0(z) \prod_{i=1}^m \left| \frac{z - z_i}{\Phi_R(z) B_R^i(z)} \right|^{\gamma_i} |P_n(z)|^p d\sigma_z \end{aligned}$$

From (28) and (30) we get

$$\begin{aligned} & \iint_{G_R \setminus G} h(z) |P_n(z)|^p d\sigma_z \\ & \prec \prod_{i=1}^m \left[ \frac{\max_{z \in \overline{G_R \setminus G}} |\Phi_R(z) B_R^i(z)|}{\min_{z \in \overline{G \setminus G^*}} |\Phi_R(z) B_R^i(z)|} \right]^{\gamma_i} \rho_2^{pn+2} \iint_{G \setminus G^*} h(z) |P_n(z)|^p d\sigma_z \end{aligned} \quad (32)$$

Since

$$\begin{aligned} |\Phi_R(z) B_R^i(z)| &= \left| \Phi_R(z) \cdot \frac{\Phi_R(z) - \Phi_R(z_i)}{\frac{1}{\overline{\Phi_R(z_i)}} - \Phi_R(z)} \cdot \frac{1}{\overline{\Phi_R(z_i)}} \right| \\ &= \left| \frac{\Phi_R(z)}{\overline{\Phi_R(z_i)}} \right| \cdot \left| \frac{\Phi_R(z) - \Phi_R(z_i)}{\Phi_R(z_i) - \Phi_R(z)} \right| = \left| \frac{\Phi_R(z)}{\overline{\Phi_R(z_i)}} \right|, \end{aligned}$$

from (32) we obtain

$$\begin{aligned} & \iint_{G_R \setminus G} h(z) |P_n(z)|^p d\sigma_z \\ & \prec \prod_{i=1}^m \left[ \frac{\max_{z \in \overline{G_R \setminus G}} |\Phi_R(z)|}{\min_{z \in \overline{G \setminus G^*}} |\Phi_R(z)|} \right]^{\gamma_i} \rho_2^{pn+2} \iint_{G \setminus G^*} h(z) |P_n(z)|^p d\sigma_z \\ & \prec \rho_2^{pn+2} \iint_{G \setminus G^*} h(z) |P_n(z)|^p d\sigma_z. \end{aligned}$$

Since  $\rho_2$  and  $R$  is symmetric, the proof is completed.

## 4 Case of Arbitrary Polynomials.

Let  $P_n(z)$  be arbitrary polynomial of degree at most  $n$ ;  $M_{n,p} := \|P_n\|_{A_p(h,G)}$ .

**Theorem 10** *Let  $p > 1$ ;  $\Omega \in Q_\alpha(\nu_1)$ , for some  $0 < \nu_1 < 1$  and  $\alpha(2-\nu_1) \geq 1$ ;  $h(z)$  defined by (1). If*

$$1 + \frac{\gamma_1}{2} < \frac{1}{\alpha(2-\nu_1)},$$

*then for every  $z \in \overline{G}$  and each  $n = 1, 2, \dots$*

$$|P_n(z)| \leq \left( c_1 n^{s^1} + c_2 |z - z_1|^{\sigma^1} n^{\frac{2}{p\alpha}} \right) M_{n,p}, \quad (33)$$

where

$$s^1 = \frac{(2 + \gamma_1)(2 - \nu_1)}{p}, \quad \sigma^1 = \frac{2}{p\alpha(2 - \nu_1)} - \frac{2 + \gamma_1}{p}. \quad (34)$$

**Proof.** Since  $L$  is a quasicircle, then any  $L_R$ ,  $n = 1 + cn^{-1}$ , also is a quasicircle. Therefore, we can construct reflection  $y_R$ ,  $y_R(0) = \infty$ , across  $L_R$  such that it satisfied the conditions (12) described for  $y_R(\zeta)$ . For this  $y_R(\zeta)$ , we can write for  $P_n(z)$  the following integral representation [7, p.105]

$$P_n(z) = -\frac{1}{\pi} \iint_{G_R} \frac{P_n(\zeta) y_{R,\bar{\zeta}}(\zeta)}{(y_R(\zeta) - z)^2} d\sigma_\zeta, \quad z \in G_R. \quad (35)$$

For  $\varepsilon > 0$  by setting  $U_\varepsilon(z) := \{\zeta : |\zeta - z| < \varepsilon\}$  and without loss of generality we may take  $U_\varepsilon := U_\varepsilon(0) \subset G^*$ . For  $z_1 \in L$  we have

$$|P_n(z_1)| \leq \frac{1}{\pi} \left\{ \iint_{U_\varepsilon} + \iint_{G_R \setminus U_\varepsilon} \right\} \frac{|P_n(\zeta)| |y_{R,\bar{\zeta}}(\zeta)|}{|y_R(\zeta) - z_1|^2} d\sigma_\zeta =: J_1 + J_2 \quad (36)$$

To estimate the integral  $J_1$ , we multiply the numerator and denominator of integrant to  $h^{1/p}(\zeta)$ , and applying the Holder inequality we get

$$J_1 \prec \left\{ \iint_{U_\varepsilon} h(\zeta) |P_n(\zeta)|^p d\sigma_\zeta \right\}^{\frac{1}{p}} \left\{ \iint_{U_\varepsilon} \frac{|y_{R,\bar{\zeta}}(\zeta)|^q}{h^{q-1}(\zeta) |y_R(\zeta) - z_1|^{2q}} d\sigma_\zeta \right\}, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

The first multiplier smaller than  $\pi^{-1}M_{n,p}$ . According to (12)  $|y_{R,\bar{\zeta}}| \asymp |y_R(\zeta)|^2$ , for all  $\zeta \in U_\varepsilon$ , because of  $|\zeta - z_1| \geq \varepsilon$ ,  $|y_R(\zeta) - z| \asymp |y_R(\zeta)|$  for  $z \in L$  and  $\zeta \in U_\varepsilon$ , then we can find

$$J_1 \prec M_{n,p} \quad (37)$$

If  $\mathcal{L}_{y_R} := |y_{R,\zeta}|^2 - |y_{R,\bar{\zeta}}|^2$  Jacobian of the reflection  $y_R(\zeta)$ , we can obtain

$$|\mathcal{L}_{y_R}| \succ |y_{R,\bar{\zeta}}|^2 \quad (38)$$

as it is in [2]. Then for the  $J_2$  we get

$$\begin{aligned} J_2 &\prec M_{n,p} \left\{ \iint_{G_R \setminus U_\varepsilon} \frac{d\sigma_\zeta}{|\zeta - z_1|^{\gamma(q-1)} |y_R(\zeta) - z_1|^{2q}} \right\}^{\frac{1}{q}} \\ &\prec M_{n,p} \left\{ \iint_{y_R(G_R \setminus U_\varepsilon)} \frac{d\sigma_\zeta}{|y_R(\zeta) - z_1|^{\gamma(q-1)} |\zeta - z_1|^{2q}} \right\}^{\frac{1}{q}}, \end{aligned} \quad (39)$$

from (12),(38) and Lemma 9.

First of all we shall establish that

$$|\zeta - z_1| \prec |y_R(\zeta) - z_1| \quad (40)$$

for all  $\zeta \in G_R \setminus U_\varepsilon$  and  $z_1 \in L$ .

In fact, let  $|z_1 - t| = d(z_1, L_R)$ ,  $t \in L_R$ . According by (12)

$$c_1 |\zeta - z| \leq |y_R(\zeta) - z| \leq c_2 |\zeta - z|, \quad (41)$$

for all  $\zeta \in G_R \setminus U_\varepsilon$  and  $z \in L_R$ , then

$$\begin{aligned} |\zeta - z_1| &\leq |\zeta - t| + |y_R(\zeta) - t| + |y_R(\zeta) - z_1| \\ &\leq (c_1^{-1} + 1) |y_R(\zeta) - t| + |y_R(\zeta) - z_1| \\ &\prec |y_R(\zeta) - z_1|. \end{aligned}$$

If  $\gamma_1 \leq 0$ , according to (40), after the carrying out that the change of variable  $\zeta = y_R(\zeta)$ , and using (38) and (12), from (39) we have

$$\begin{aligned} J_2 &\prec M_{n,p} \left\{ \iint_{G_R \setminus U_\varepsilon} \frac{d\sigma_\zeta}{|y_R(\zeta) - z_1|^{\gamma_1(q-1)+2q}} \right\}^{\frac{1}{q}} \\ &\prec M_{n,p} \left\{ \iint_{y_R(G_R \setminus U_\varepsilon)} \frac{d\sigma_\zeta}{|\zeta - z_1|^{\gamma_1(q-1)+2q}} \right\}^{\frac{1}{q}} \\ &\prec M_{n,p} d^{-\frac{2+\gamma_1}{p}}(z_1, L_R) \end{aligned} \quad (42)$$

If  $\gamma_1 > 0$ , by changing the variable  $\zeta = y_R(\zeta)$  and applying (40), (38) and (12) we obtain

$$J_2 \prec M_{n,p} \left\{ \iint_{y_R(G_R \setminus U_\varepsilon)} \frac{d\sigma_\zeta}{|\zeta - z_1|^{\gamma_1(q-1)+2q}} \right\}^{\frac{1}{q}} \prec M_{n,p} d^{-\frac{2+\gamma_1}{p}}(z_1, L_R) \quad (43)$$

From (37), (39), (42) and (43) we obtain

$$|P_n(z_1)| \prec M_{n,p} d^{-\frac{2+\gamma_1}{p}}(z_1, L_R).$$

Since  $\Omega \in Q_\alpha(\nu_1)$ , then  $\Psi \in Lip \frac{1}{2-\nu_1}$  [11] and so

$$|P_n(z_1)| \prec M_{n,p} n^{\frac{(2+\gamma_1)(2-\nu_1)}{p}} \quad (44)$$

Now, using the integral representation (35) we have

$$\begin{aligned} \left| \frac{P_n(z) - P_n(z_1)}{(z - z_1)^{\sigma^1}} \right| &\leq \frac{1}{\pi} \iint_{G_R} \frac{|P_n(\zeta)| |y_{R,\bar{\zeta}}(\zeta)| |z - z_1|^{1-\sigma^1}}{|y_R(\zeta) - z| |y_R(\zeta) - z_1|^2} d\sigma_\zeta \\ &+ \frac{1}{\pi} \iint_{G_R} \frac{|P_n(\zeta)| |y_{R,\bar{\zeta}}(\zeta)| |z - z_1|^{1-\sigma^1}}{|y_R(\zeta) - z_1| |y_R(\zeta) - z|^2} d\sigma_\zeta =: A(z; z_1) + B(z; z_1). \end{aligned} \quad (45)$$

From definitions of the integrals  $A(z; z_1)$  and  $B(z; z_1)$  we see that they are symmetric respect to the points  $z$  and  $z_1$ . Thus, we will estimate integrals  $A(z; z_1)$  and  $B(z; z_1)$  parallel. To estimate the integral  $A(z; z_1)$  ( $B(z; z_1)$ ) we multiply the numerator and denominator of integrant to  $|\zeta - z_1|^{\gamma_1/p}$ , and applying the Holder inequality from Lemma 9 we get

$$\begin{aligned} A(z; z_1) &\prec M_{n,p} \left\{ \left( \iint_{U_\varepsilon} + \iint_{G_R \setminus U_\varepsilon} \right) \frac{|y_{R,\bar{\zeta}}(\zeta)| |z - z_1|^{q(1-\sigma^1)} d\sigma_\zeta}{|\zeta - z_1|^{\gamma_1(q-1)} |y_R(\zeta) - z|^q |y_R(\zeta) - z_1|^{2q}} \right\}^{1/q} \\ &=: M_{n,p} \{A_1(z; z_1) + A_2(z; z_1)\}^{1/q}, \quad \frac{1}{p} + \frac{1}{q} = 1; \end{aligned} \quad (46)$$

$$\begin{aligned} B(z; z_1) &\prec M_{n,p} \left\{ \left( \iint_{U_\varepsilon} + \iint_{G_R \setminus U_\varepsilon} \right) \frac{|y_{R,\bar{\zeta}}(\zeta)| |z - z_1|^{q(1-\sigma^1)} d\sigma_\zeta}{|\zeta - z_1|^{\gamma_1(q-1)} |y_R(\zeta) - z|^{2q} |y_R(\zeta) - z_1|^q} \right\}^{1/q} \\ &\prec M_{n,p} \{B_1(z; z_1) + B_2(z; z_1)\}^{1/q} \end{aligned}$$

According to (12)  $|y_{R,\bar{\zeta}}(\zeta)| \asymp |y_R(\zeta)|^2$ , for all  $\zeta \in U_\varepsilon$ , because of  $|\zeta - z_1| \asymp 1$ ,  $|y_R(\zeta)| \asymp |y_R(\zeta) - z| \asymp |y_R(\zeta) - z_1|$  for  $z, z_1 \in L$  and  $\zeta \in U_\varepsilon$ , then we can find

$$A_1(z; z_1) \prec 1 \quad (B_1(z; z_1) \prec 1) \quad (47)$$

For the estimations of the  $A_2(z; z_1)$  ( $B_2(z; z_1)$ ) we consider the different situations of the points  $z$  and  $z_1$  on  $L$ . Let us set

$$\begin{aligned} F_1 &:= y_R(G_R \setminus U_\varepsilon) = E_0 := E_1 \cup E_2 \\ F_{11} &:= \left\{ \zeta \in F_1 : |\zeta - z_1| \leq \frac{1}{2} |z - z_1| \right\}; F_{11}^c := \left\{ \zeta \in F_1 : |\zeta - z_1| > \frac{1}{2} |z - z_1| \right\}; \\ F_{12} &:= \left\{ \zeta \in F_1 : |\zeta - z| \leq \frac{1}{2} |z - z_1| \right\}; F_{12}^c := \left\{ \zeta \in F_1 : |\zeta - z| > \frac{1}{2} |z - z_1| \right\}. \\ E_1 &:= \{y_R(G_R \setminus U_\varepsilon) \cap U_\delta(z_1)\}, E_2 := \{y_R(G_R \setminus U_\varepsilon) \setminus U_\delta(z_1)\}, 0 < \delta < \delta_0(G); \\ E_{01} &:= \{\zeta \in E_0 : |\zeta - z_1| \geq |\zeta - z|\}, E_{02} := \{\zeta \in E_0 : |\zeta - z_1| < |\zeta - z|\}. \end{aligned}$$

a) Let  $|z - z_1| \geq \delta > 0$ .

Taking into account (12) (for the  $y_R$ ) and (40) we have

$$\begin{aligned} A_2(z; z_1) &\prec \iint_{F_1} \frac{d\sigma_\zeta}{|y_R(\zeta) - z_1|^{\gamma_1(q-1)} |\zeta - z|^q |\zeta - z_1|^{2q}} \\ &\leq \left( \iint_{F_{11}} + \iint_{F_{12}} + \iint_{F_{11}^c} + \iint_{F_{12}^c} \right) \frac{d\sigma_\zeta}{|y_R(\zeta) - z_1|^{\gamma_1(q-1)} |\zeta - z|^q |\zeta - z_1|^{2q}} \end{aligned}$$

According to  $|\zeta - z| \geq ||z - z_1| - |\zeta - z_1|| \geq \frac{1}{2} |z - z_1|$ , for  $\zeta \in F_{11}$  and  $|\zeta - z_1| \geq \frac{1}{2} |z - z_1|$ , for  $\zeta \in F_{12}$ , we obtain

$$\begin{aligned} &\left( \iint_{F_{11}} + \iint_{F_{12}} \right) \frac{d\sigma_\zeta}{|y_R(\zeta) - z_1|^{\gamma_1(q-1)} |\zeta - z|^q |\zeta - z_1|^{2q}} \\ &\prec \iint_{F_{11}} \frac{d\sigma_\zeta}{|\zeta - z_1|^{\gamma_1(q-1)+2q}} + \iint_{F_{12}} \frac{d\sigma_\zeta}{|\zeta - z|^q} \\ &\prec n^{(\gamma_1+2)(2-\nu_1)(q-1)} + n^{\frac{q-2}{\alpha}}; \\ &\iint_{F_{11}^c} + \iint_{F_{12}^c} \frac{d\sigma_\zeta}{|y_R(\zeta) - z_1|^{\gamma_1(q-1)} |\zeta - z|^q |\zeta - z_1|^{2q}} \prec 1, \end{aligned}$$

and

$$A_2(z; z_1) \prec n^{(\gamma_1+2)(2-\nu_1)(q-1)} + n^{\frac{q-2}{\alpha}} \quad (48)$$

Analogously,

$$B_2(z; z_1) \prec \iint_{F_1} \frac{d\sigma_\zeta}{|y_R(\zeta) - z_1|^{\gamma_1(q-1)} |\zeta - z|^{2q} |\zeta - z_1|^q}$$

$$\leq \left( \iint_{F_{11}} + \iint_{F_{12}} + \iint_{F_{11}^c} + \iint_{F_{12}^c} \right) \frac{d\sigma_\zeta}{|y_R(\zeta) - z_1|^{\gamma_1(q-1)} |\zeta - z|^{2q} |\zeta - z_1|^q}.$$

Since

$$\begin{aligned} & \left( \iint_{F_{11}} + \iint_{F_{12}} \right) \frac{d\sigma_\zeta}{|y_R(\zeta) - z_1|^{\gamma_1(q-1)} |\zeta - z|^{2q} |\zeta - z_1|^q} \\ & \prec \iint_{F_{11}} \frac{d\sigma_\zeta}{|\zeta - z_1|^{\gamma_1(q-1)+q}} + \iint_{F_{12}} \frac{d\sigma_\zeta}{|\zeta - z|^{2q}} \\ & \prec n^{(\gamma_1(q-1)+q-2)(2-\nu_1)} + n^{\frac{2(q-1)}{\alpha}}; \end{aligned}$$

and

$$\iint_{F_{11}^c} + \iint_{F_{12}^c} \frac{d\sigma_\zeta}{|y_R(\zeta) - z_1|^{\gamma_1(q-1)} |\zeta - z|^{2q} |\zeta - z_1|^q} \prec 1,$$

then

$$B_2(z; z_1) \prec n^{(\gamma_1(q-1)+q-2)(2-\nu_1)} + n^{\frac{2(q-1)}{\alpha}} \quad (49)$$

From (46), (47), (48) and (49) we get

$$A(z; z_1) \prec M_{n,p} n^{\frac{2}{\alpha p}}, \quad B(z; z_1) \prec M_{n,p} n^{\frac{2}{\alpha p}}. \quad (50)$$

b) Let  $\delta > |z - z_1| \geq d(z_1, L_R)$ .

Taking into account that  $|z - z_1|^\varepsilon \leq c(\varepsilon) (|\zeta - z|^\varepsilon + |\zeta - z_1|^\varepsilon)$  satisfies for all  $\varepsilon > 0$ , we have

$$\begin{aligned} A_2(z; z_1) & \prec \iint_{F_1} \frac{d\sigma_\zeta}{|\zeta - z_1|^{\gamma_1(q-1)+2q} |\zeta - z|^{q\sigma^1}} \\ & \quad + \iint_{F_1} \frac{d\sigma_\zeta}{|\zeta - z_1|^{\gamma_1(q-1)+2q-(1-\sigma^1)q} |\zeta - z|^q} \\ & \leq \iint_{E_{01}} \frac{d\sigma_\zeta}{|\zeta - z|^{\gamma_1(q-1)+2q+q\sigma^1}} + \iint_{E_{01}} \frac{d\sigma_\zeta}{|\zeta - z|^{\gamma_1(q-1)+2q-(1-\sigma^1)q+q}} \\ & \quad + \iint_{E_{02}} \frac{d\sigma_\zeta}{|\zeta - z_1|^{\gamma_1(q-1)+2q+q\sigma^1}} + \iint_{E_{02}} \frac{d\sigma_\zeta}{|\zeta - z_1|^{\gamma_1(q-1)+2q-(1-\sigma^1)q+q}} \\ & \prec n^{((\gamma_1+2)(q-1)+q\sigma^1)(2-\nu_1)} \quad (51) \end{aligned}$$



Entirely analogously we see that

$$B_2(z; z_1) \prec n^{((\gamma_1+2)(q-1)+q\sigma^1)(2-\nu_1)} \quad (52)$$

Therefore, in this case, according (51) and (52) in (46) from (47) and (34) we get

$$A(z; z_1) \prec M_{n,p} n^{(\gamma_1+2+\sigma^1)(2-\nu_1)} \prec M_{n,p} n^{\frac{2}{p\alpha}} \quad (53)$$

and, respectively

$$B(z; z_1) \prec M_{n,p} n^{(\gamma_1+2+\sigma^1)(2-\nu_1)} \prec M_{n,p} n^{\frac{2}{p\alpha}} \quad (54)$$

c) Let  $|z - z_1| \leq d(z_1, L_R)$ .

From (46) we have

$$A_2(z; z_1) \prec \iint_{G_R \setminus U_\varepsilon} \frac{d^{q(1-\sigma^1)}(z_1, L_R) d\sigma_\zeta}{d^{\gamma_1(q-1)+3q}(z_1, L_R)} \prec n^{((\gamma_1+2)(q-1)+q\sigma^1)(2-\nu_1)}, \quad (55)$$

and, respectively

$$B_2(z; z_1) \prec n^{((\gamma_1+2)(q-1)+q\sigma^1)(2-\nu_1)}. \quad (56)$$

According (55) and (56) in (46) from 47) and (34) we obtain

$$A(z; z_1) \prec M_{n,p} n^{\frac{2}{p\alpha}}, \quad B(z; z_1) \prec M_{n,p} n^{\frac{2}{p\alpha}} \quad (57)$$

So, from (57), (44) and (45) we obtain the proof of (33).

**Theorem 11** Let  $p > 1$ ;  $\Omega \in Q_\alpha(\nu_1, \dots, \nu_m)$ , for some  $0 < \nu_i < 1$  and  $\alpha(2 - \nu_i) \geq 1$ ,  $h(z)$  defined by (1). If

$$1 + \frac{\gamma_i}{2} > \frac{1}{\alpha(2 - \nu_i)}$$

is satisfied for any points  $z_i \in L$ ,  $i = \overline{1, m}$  Then, for every  $n = 1, 2, \dots$

$$\max_{z \in \overline{G}} \left( \prod_{i=1}^m |z - z_i|^{\tilde{\mu}_i} |P_n(z)| \right) \leq c_4 n^{1/\alpha} M_{n,p},$$

$$|P_n(z_i)| \leq c_5 n^{\tilde{s}_i} M_{n,p},$$

where

$$\hat{\mu}_i := \frac{2 + \gamma_i}{p} - \frac{2}{p\alpha(2 - \nu_i)}, \quad \hat{s}_i := \frac{(2 + \gamma_i)(2 - \nu_i)}{p}, \quad i = \overline{1, m}.$$

The proof of this theorem follows from one of the general theorem of the work [4].

## 5 Proof of Theorem's 5, 6.

Since  $M_{n,2} \equiv 1$  for the  $K_n(z)$ , then we get the proof of Theorem 5 and Theorem 6 from Theorem 10 and Theorem 11 respectively.

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## Dual Riesz Bases and the Canonical Operator

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**Abstract.** Let  $g_{j,k}(\mathbf{x}) := 2^{nj/2}g(2^j\mathbf{x} - \mathbf{k})$ . A set  $G_0 := \{g^\ell, \ell = 1, \dots, m\}$  of functions in  $L^2(\mathbb{R}^n)$  is called an  $R$ -family if  $G := \{g_{j,\mathbf{k}}^\ell; \ell = 1, \dots, m, j \in \mathbb{Z}, \mathbf{k} \in \mathbb{Z}^n\}$  is a Riesz basis of  $L^2(\mathbb{R}^n)$ . If both  $G$  and its dual are generated by  $R$ -families, then  $G_0$  is called a  $W$ -family. In this article we present conditions under which a Riesz basis is generated by a  $W$ -family. The main result is a method to obtain  $W$ -families generated by multiresolution analyses by perturbations of semiorthogonal  $W$ -families generated by multiresolution analyses. As an application we give examples of affine Riesz bases that are not semiorthogonal, but are generated by  $W$ -families.

**Keywords:** Riesz bases, dual Riesz bases, wavelets,  $R$ -families,  $W$ -families, multiresolution analysis, semiorthogonal wavelets.

### 1. INTRODUCTION

In the sequel  $\mathbb{Z}$  will denote the integers,  $\mathbb{Z}^+$  the strictly positive integers,  $\mathbb{Z}_0$  the nonnegative integers,  $\mathbb{R}$  the real numbers, and  $\mathbb{C}$  the complex numbers. Unless otherwise indicated,  $t$ ,  $x$ , and  $\omega$  will denote real variables,  $\mathbf{x}$  will denote an element of  $\mathbb{R}^n$ , and  $z$  will denote a complex variable.  $I$  will stand for the identity operator. Given a bounded linear operator  $A$ , its (Hilbert space) adjoint will be denoted by  $A^*$ .

Unless otherwise indicated, the following definitions and basic properties may be found in, e. g., [17].

Let  $\mathcal{H}$  be a (separable) Hilbert space with inner product  $\langle \cdot, \cdot \rangle$  and norm  $\|\cdot\| := \langle \cdot, \cdot \rangle^{1/2}$ . A sequence  $\mathbb{F} := \{f_n, n \in \mathbb{Z}^+\} \subset \mathcal{H}$  is called a *frame* if there are constants  $A$  and  $B$  such that for every  $f \in \mathcal{H}$

$$A\|f\|^2 \leq \sum_{n \in \mathbb{Z}^+} |\langle f, f_n \rangle|^2 \leq B\|f\|^2.$$

The constants  $A$  and  $B$  are called *bounds* of the frame.

If only the right-hand inequality is satisfied for all  $f \in \mathcal{H}$ , then  $\mathbb{F}$  is called a *Bessel sequence* with bound  $B$ .

A frame is called *exact*, or a *Riesz basis*, if upon the removal of any single element of the sequence, it ceases to be a frame. A sequence  $\mathbb{F} \subset \mathcal{H}$  is a Riesz basis if and only if it is complete, and its moment space is  $\ell^2$ , i. e., if for any sequence  $\{\alpha_k\} \in \ell^2$  there is an element  $f \in \mathcal{H}$  such that  $\langle f, f_k \rangle = \alpha_k$ .

It is also known that  $\mathbb{F}$  is a Riesz basis if and only if it is complete, and there exist strictly positive constants  $A$  and  $B$  such that for any  $n \in \mathbb{Z}^+$  and arbitrary scalars  $c_1, \dots, c_n$ ,

$$(1.1) \quad A \sum_{k=1}^n |c_k|^2 \leq \left\| \sum_{k=1}^n c_k f_k \right\|^2 \leq B \sum_{k=1}^n |c_k|^2.$$

The constants  $A$  and  $B$  are called *Riesz bounds*. For a Riesz basis, frame bounds and Riesz bounds coincide. If we say that  $\mathbb{F}$  is a Riesz basis with bounds  $A$  and  $B$ , we mean that  $A$  and  $B$  are its frame (or Riesz) bounds.

Another characterization is the following: a sequence  $\mathbb{F} \subset \mathcal{H}$  is a Riesz basis if and only if it is the image of an orthonormal basis under a bounded invertible linear operator.

A sequence  $\mathbb{G} \subset \mathcal{H}$  is *biorthogonal* to  $\mathbb{F}$  if

$$\langle f_k, g_n \rangle = \delta_{kn}, \quad k, n \in \mathbb{Z}^+.$$

The operator  $U : \mathcal{H} \rightarrow \mathcal{H}$  defined by

$$Uf := \sum_{n=1}^{\infty} \langle f, f_n \rangle f_n$$

is self-adjoint, positive-definite and invertible (cf. [8]). If  $\tilde{f}_n := U^{-1}f_n$ , the sequence  $\tilde{\mathbb{F}} := \{\tilde{f}_n, n \in \mathbb{Z}^+\}$  is a frame with bounds  $B^{-1}$  and  $A^{-1}$ , called the *dual frame*. Moreover, if  $\mathbb{F}$  is a Riesz basis, then also  $\tilde{\mathbb{F}}$  is a Riesz basis. From e. g. [1, Theorem 3.5(e)] we deduce that if  $\mathbb{F}$  is a Riesz basis, then  $\tilde{\mathbb{F}}$  is the only sequence in  $\mathcal{H}$  that is biorthogonal to  $\mathbb{F}$ .

If  $\mathbb{F}$  is a Riesz basis, then any  $f \in \mathcal{H}$  has the following representations:

$$f = \sum_{n=1}^{\infty} \langle f, \tilde{f}_n \rangle f_n = \sum_{n=1}^{\infty} \langle f, f_n \rangle \tilde{f}_n.$$

In the sequel  $n \in \mathbb{Z}^+$  will be arbitrary but fixed, and  $\mathbf{x} \in \mathbb{R}^n$ . Given a function  $f$  with domain  $\mathbb{R}^n$  we define  $f_{j,k}(\mathbf{x}) := 2^{nj/2} f(2^j \mathbf{x} - \mathbf{k})$ . We will use the following notation:  $\Psi := \{\psi_{j,\mathbf{k}}^\ell; \ell = 1, \dots, m, j \in \mathbb{Z}, \mathbf{k} \in \mathbb{Z}^n\}$ ,  $\mathbb{G} := \{g_{j,\mathbf{k}}^\ell; \ell = 1, \dots, m, j \in \mathbb{Z}, \mathbf{k} \in \mathbb{Z}^n\}$ ,  $\mathbb{H} := \{h_{j,\mathbf{k}}^\ell; \ell = 1, \dots, m, j \in \mathbb{Z}, \mathbf{k} \in \mathbb{Z}^n\}$ , and  $V := \{v(\cdot, \ell, j, \mathbf{k}); \ell = 1, \dots, m, j \in \mathbb{Z}, \mathbf{k} \in \mathbb{Z}^n\}$ .

Following [3], a family  $\underline{g} := \{g^\ell, \ell = 1, \dots, m\}$  of  $m$  functions in  $L^2(\mathbb{R}^n)$  will be called an  $R$ -family if  $\mathbb{G}$  is a Riesz basis of  $L^2(\mathbb{R}^n)$ .  $\mathbb{V}$  will be said to be generated by  $\underline{v} := \{v^\ell, \ell = 1, \dots, m\}$  if  $v(\cdot, \ell, j, \mathbf{k}) = v_{j,\mathbf{k}}^\ell(\cdot)$ ,  $\ell = 1, \dots, m, j \in \mathbb{Z}, \mathbf{k} \in \mathbb{Z}^n$ . Clearly  $\underline{v}$  is an  $R$ -family. Although  $\mathbb{V}$  may be generated by an  $R$ -family, its dual basis may not be generated by an  $R$ -family (cf., eg. [2, 4, 15]). If  $\mathbb{V}$  is generated by  $\underline{v}$  and also its dual is generated by an  $R$ -family, then  $\underline{v}$  will be called a  $W$ -family.

The main objective of this paper is to find conditions under which a Riesz basis  $\mathbb{V}$  is generated by a  $W$ -family. We find these conditions by choosing an arbitrary Riesz basis  $\mathbb{G}$  of  $L^2(\mathbb{R}^n)$  generated by a  $W$ -family  $\underline{g}$  and studying the properties of the *canonical operator* from  $\mathbb{G}$  to  $\mathbb{V}$ , i. e., the bounded invertible linear operator from  $L^2(\mathbb{R}^n)$  onto  $L^2(\mathbb{R}^n)$ , that maps the functions  $g_{j,k}^\ell$  onto the functions  $v(\cdot, \ell, j, \mathbf{k})$ . Chui and Shi have shown that semiorthogonal wavelets in  $\mathbb{R}$  are generated by  $W$ -families ([3]). In this paper we develop a method to obtain  $W$ -families generated by multiresolution analyses by perturbations of semiorthogonal  $W$ -families generated by multiresolution analyses. As an application we give examples of affine Riesz bases that are not semiorthogonal, but are generated by  $W$ -families.

## 2. CANONICAL OPERATORS AND RIESZ BASES

**Lemma 2.1.** *Let  $\mathbb{E} := \{e_j, j \in \mathbb{Z}^+\}$  be a Riesz basis with bounds  $A_1$  and  $B_1$  for a Hilbert space  $(\mathcal{H}_1, \langle \cdot, \cdot \rangle_1)$  and let  $\mathbb{F} := \{f_j, j \in \mathbb{Z}^+\}$  be a Bessel sequence with bound  $B$  for a Hilbert space  $(\mathcal{H}_2, \langle \cdot, \cdot \rangle_2)$ . Then there exists a unique bounded linear operator  $S$  from  $\mathcal{H}_1$  into  $\mathcal{H}_2$ , such that  $Se_j = f_j, j \in \mathbb{Z}^+$ . Moreover,  $\|S\| \leq \sqrt{B/A_1}$ .*

*Proof.* Let  $h : \mathcal{H}_1 \times \mathcal{H}_2 \rightarrow \mathbb{C}$  be defined by

$$h(x, y) := \sum_{j \in \mathbb{Z}^+} \langle x, \tilde{e}_j \rangle_1 \langle f_j, y \rangle_2.$$

Then, applying the Cauchy–Schwarz inequality, we have:

$$\begin{aligned} |h(x, y)| &= \left| \sum_{j \in \mathbb{Z}^+} \langle x, \tilde{e}_j \rangle_1 \langle f_j, y \rangle_2 \right| \leq \left( \sum_{j \in \mathbb{Z}^+} |\langle x, \tilde{e}_j \rangle_1|^2 \right)^{1/2} \left( \sum_{j \in \mathbb{Z}^+} |\langle f_j, y \rangle_2|^2 \right)^{1/2} \\ &\leq \sqrt{1/A_1} \|x\|_{\mathcal{H}_1} \sqrt{B} \|y\|_{\mathcal{H}_2}. \end{aligned}$$

Hence  $h$  is a bounded sesquilinear form, and from the Riesz representation theorem ([13]) there exists a bounded linear operator  $S : \mathcal{H}_1 \rightarrow \mathcal{H}_2$  such that  $h(u_1, u_2) = \langle Su_1, u_2 \rangle_2$  and  $\|S\| = \|h\|$ .

Note that

$$h(e_j, u_2) = \sum_{k \in \mathbb{Z}^+} \langle e_j, \tilde{e}_k \rangle_1 \langle f_k, u_2 \rangle_2 = \sum_{k \in \mathbb{Z}^+} \delta_{j,k} \langle f_k, u_2 \rangle_2 = \langle f_j, u_2 \rangle_2.$$

Hence  $\langle Se_j, u_2 \rangle_2 = \langle f_j, u_2 \rangle_2$  for every  $u_2 \in \mathcal{H}_2$ , and we deduce that  $Se_j = f_j$ ,  $j \in \mathbb{Z}^+$ .

Since  $|h(x, y)| \leq \sqrt{B/A_1} \|x\|_{\mathcal{H}_1} \|y\|_{\mathcal{H}_2}$ , we see that

$$\|h\| = \sup\{|h(x, y)|, x \in \mathcal{H}_1, y \in \mathcal{H}_2, \|x\| = 1, \|y\| = 1\} \leq \sqrt{B/A_1}.$$

Since  $\{e_j, j \in \mathbb{Z}^+\}$  is a basis, the uniqueness of  $S$  follows.  $\square$

The operator  $S$  will be called the *canonical operator* from  $\mathbb{E}$  onto  $\mathbb{F}$ . As a consequence of Lemma 2.1 we obtain:

**Lemma 2.2.** *Let  $\mathbb{E} := \{e_j, j \in \mathbb{Z}^+\}$  be a Riesz basis with bounds  $A_1$  and  $B_1$  for a Hilbert space  $(\mathcal{H}_1, \langle \cdot, \cdot \rangle_1)$  and let  $\mathbb{F} := \{f_j, j \in \mathbb{Z}^+\}$  be a Riesz basis with bounds  $A$  and  $B$  for a Hilbert space  $(\mathcal{H}_2, \langle \cdot, \cdot \rangle_2)$ . If  $S$  denotes the canonical operator from  $\mathbb{E}$  onto  $\mathbb{F}$ , then  $\|S\| \leq \sqrt{B/A_1}$  and  $\|S^{-1}\| \leq \sqrt{B_1/A}$ .*

*Proof.* The bound for  $\|S\|$  follows from Lemma 2.1.

Let  $f \in \mathcal{H}_2$ . Since  $\mathbb{F}$  is a Riesz basis,

$$f = \sum_{j \in \mathbb{Z}^+} c_j f_j, \quad \{c_j\} \in \ell^2.$$

Thus

$$S^{-1}f = \sum_{j \in \mathbb{Z}^+} c_j S^{-1}f_j = \sum_{j \in \mathbb{Z}^+} c_j e_j.$$

This implies that

$$\|S^{-1}f\|^2 = \left\| \sum_{j \in \mathbb{Z}^+} c_j e_j \right\|^2 \leq B_1 \sum_{j \in \mathbb{Z}^+} |c_j|^2.$$

But

$$A \sum_{j \in \mathbb{Z}^+} |c_j|^2 \leq \left\| \sum_{j \in \mathbb{Z}^+} c_j f_j \right\|^2 = \|f\|^2.$$

Thus  $\sum_{j \in \mathbb{Z}^+} |c_j|^2 \leq (1/A) \|f\|^2$ , and the assertion follows.  $\square$

In the sequel  $D : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$  will denote the dilation operator:

$$Df(\mathbf{x}) := 2^{n/2}f(2\mathbf{x}).$$

Clearly

$$D^j f(\mathbf{x}) = 2^{nj/2}f(2^j\mathbf{x}), \quad j \in \mathbb{Z}, \mathbf{x} \in \mathbb{R}^n.$$

Let  $\mathbf{k} = (k_1, \dots, k_n) \in \mathbb{Z}^n$ ,  $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$ ,  $f \in L^2(\mathbb{R}^n)$ , and

$$T_{x_i} f(\mathbf{x}) := f(x_1, \dots, x_{i-1}, x_i - 1, x_{i+1}, \dots, x_n).$$

Then

$$T_{x_i}^{k_i} f(\mathbf{x}) = f(x_1, \dots, x_{i-1}, x_i - k_i, x_{i+1}, \dots, x_n).$$

Setting

$$T^{\mathbf{k}} := T_{x_1}^{k_1} \dots T_{x_n}^{k_n} := \prod_{i=1}^n T_{x_i}^{k_i}$$

we have  $T^{\mathbf{k}} : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$  and

$$T^{\mathbf{k}} f(\mathbf{x}) = f(\mathbf{x} - \mathbf{k}), \quad f \in L^2(\mathbb{R}^n), \quad \mathbf{k} \in \mathbb{Z}^n, \mathbf{x} \in \mathbb{R}^n.$$

The following two propositions were inspired by the discussion in [6, p. 70]:

**Lemma 2.3.** *Let  $\Psi$  be a Riesz basis for  $L^2(\mathbb{R}^n)$  generated by a  $W$ -family, let  $\mathbb{G}$  be a Bessel sequence for  $L^2(\mathbb{R}^n)$ , and let  $S$  be the canonical operator from  $\Psi$  onto  $\mathbb{G}$ . Then, for any  $j \in \mathbb{Z}$ ,  $S$  commutes with  $D^j$ .*

*Proof.* Let  $f \in L^2(\mathbb{R}^n)$ . Since  $\Psi$  is generated by a  $W$ -family there are functions  $\tilde{\psi}_{j,\mathbf{k}}^\ell$  such that, if  $\tilde{\psi}(\cdot, \ell, j, \mathbf{k})$  denotes the adjoint of  $\psi_{j,\mathbf{k}}^\ell$ , then

$$\tilde{\psi}(\cdot, \ell, j, \mathbf{k}) = \tilde{\psi}_{j,\mathbf{k}}^\ell \quad \ell = 1, \dots, m, j \in \mathbb{Z}, \mathbf{k} \in \mathbb{Z}^n.$$

Thus

$$(2.1) \quad f = \sum_{\ell=1}^m \sum_{j \in \mathbb{Z}} \sum_{\mathbf{k} \in \mathbb{Z}^n} \langle f, \tilde{\psi}_{j,\mathbf{k}}^\ell \rangle \psi_{j,\mathbf{k}}^\ell.$$

But  $S\psi_{j,\mathbf{k}}^\ell = g_{j,\mathbf{k}}^\ell = D^j T^{\mathbf{k}} g^\ell$ . Therefore

$$(2.2) \quad Sf = \sum_{\ell=1}^m \sum_{j \in \mathbb{Z}} \sum_{\mathbf{k} \in \mathbb{Z}^n} \langle f, \tilde{\psi}_{j,\mathbf{k}}^\ell \rangle g_{j,\mathbf{k}}^\ell = \sum_{\ell=1}^m \sum_{j \in \mathbb{Z}} \sum_{\mathbf{k} \in \mathbb{Z}^n} \langle f, \tilde{\psi}_{j,\mathbf{k}}^\ell \rangle D^j T^{\mathbf{k}} g^\ell.$$

Since  $D$  is unitary, we have:

$$SDf = \sum_{\ell=1}^m \sum_{j \in \mathbb{Z}} \sum_{\mathbf{k} \in \mathbb{Z}^n} \langle Df, \tilde{\psi}_{j,\mathbf{k}}^\ell \rangle D^j T^{\mathbf{k}} g^\ell = \sum_{\ell=1}^m \sum_{j \in \mathbb{Z}} \sum_{\mathbf{k} \in \mathbb{Z}^n} \langle f, D^{-1} \tilde{\psi}_{j,\mathbf{k}}^\ell \rangle D^j T^{\mathbf{k}} g^\ell.$$

But

$$D^{-1} \tilde{\psi}_{j,\mathbf{k}}^\ell = D^{-1} D^j T^{\mathbf{k}} \tilde{\psi}^\ell = D^{j-1} T^{\mathbf{k}} \tilde{\psi}^\ell = \tilde{\psi}_{j-1,\mathbf{k}}^\ell,$$

and therefore

$$SDf = \sum_{\ell=1}^m \sum_{j \in \mathbb{Z}} \sum_{\mathbf{k} \in \mathbb{Z}^n} \langle f, \tilde{\psi}_{j-1,\mathbf{k}}^\ell \rangle D^j T^{\mathbf{k}} g^\ell.$$



This implies that

$$\begin{aligned}
 D^{-1}SDf &= \\
 \sum_{\ell=1}^m \sum_{j \in \mathbb{Z}} \sum_{\mathbf{k} \in \mathbb{Z}^n} \langle f, \tilde{\psi}_{j-1, \mathbf{k}}^\ell \rangle D^{-1}D^j T^{\mathbf{k}} g^\ell &= \sum_{\ell=1}^m \sum_{j \in \mathbb{Z}} \sum_{\mathbf{k} \in \mathbb{Z}^n} \langle f, \tilde{\psi}_{j-1, \mathbf{k}}^\ell \rangle D^{j-1} T^{\mathbf{k}} g^\ell \\
 &= \sum_{\ell=1}^m \sum_{j \in \mathbb{Z}} \sum_{\mathbf{k} \in \mathbb{Z}^n} \langle f, \tilde{\psi}_{j-1, \mathbf{k}}^\ell \rangle g_{j-1, \mathbf{k}}^\ell = \sum_{\ell=1}^m \sum_{j \in \mathbb{Z}} \sum_{\mathbf{k} \in \mathbb{Z}^n} \langle f, \tilde{\psi}_{j, \mathbf{k}}^\ell \rangle g_{j, \mathbf{k}}^\ell = Sf.
 \end{aligned}$$

Therefore  $D^{-1}SDf = Sf$ , whence  $SD = DS$ , and therefore  $SD^{-1} = D^{-1}S$ . An obvious inductive argument shows that  $SD^j = D^jS$  for every  $j \in \mathbb{Z}$ , and the conclusion follows.  $\square$

Note that  $S$  is given explicitly in (2.2).

**Lemma 2.4.** *Let  $\mathbb{E} := \{e_j, j \in \mathbb{Z}^+\}$  and  $\mathbb{F} := \{f_j, j \in \mathbb{Z}^+\}$  be Riesz bases and let  $S$  be the canonical operator from  $\mathbb{E}$  onto  $\mathbb{F}$ . If  $\tilde{\mathbb{E}} := \{\tilde{e}_j, j \in \mathbb{Z}^+\}$  is the dual Riesz basis of  $\mathbb{E}$  and  $\tilde{\mathbb{F}} := \{\tilde{f}_j, j \in \mathbb{Z}^+\}$  is the dual Riesz basis of  $\mathbb{F}$ , then  $(S^{-1})^*$  is the canonical operator from  $\tilde{\mathbb{E}}$  onto  $\tilde{\mathbb{F}}$ .*

*Proof.* Let  $U$  be the canonical operator from  $\tilde{\mathbb{E}}$  onto  $\tilde{\mathbb{F}}$ . Then

$$\delta_{i,j} = \langle \tilde{f}_i, f_j \rangle = \langle U\tilde{e}_i, Se_j \rangle = \langle \tilde{e}_i, U^*Se_j \rangle, \quad i, j \in \mathbb{Z}^+,$$

Thus

$$U^*Se_j = e_j, \quad j \in \mathbb{Z}^+.$$

Since  $\mathbb{E}$  is a basis, this implies that  $U^*S = I$ , and therefore that  $U = (S^{-1})^*$ .  $\square$

We now obtain a sufficient condition for an affine Riesz basis to be generated by a  $W$ -family.

**Theorem 2.5.** *Let  $\Psi$  be a Riesz basis for  $L^2(\mathbb{R}^n)$  generated by a  $W$ -family, let  $\mathbb{G}$  be a Riesz basis for  $L^2(\mathbb{R}^n)$ , and let  $S$  be the canonical operator from  $\Psi$  onto  $\mathbb{G}$ . Assume that the operators  $T_{x_1}, \dots, T_{x_n}$  commute with  $S$ . Then, for  $\ell = 1, \dots, m, j \in \mathbb{Z}$ , and  $\mathbf{k} \in \mathbb{Z}^n$ ,*

- (a)  $g_{j, \mathbf{k}}^\ell = (S\psi^\ell)_{j, \mathbf{k}}$ .
- (b)  $\tilde{g}(\cdot, \ell, j, \mathbf{k}) = \left( (S^{-1})^* \tilde{\psi}^\ell \right)_{j, \mathbf{k}}$ .

*Proof.* (a) From Lemma 2.3 we know that the operators  $D$  and  $S$  commute. Applying the hypothesis we have, for  $\ell = 1, \dots, m, j \in \mathbb{Z}, \mathbf{k} \in \mathbb{Z}^n$ :

$$g_{j, \mathbf{k}}^\ell = S\psi_{j, \mathbf{k}}^\ell = SD^j T^{\mathbf{k}} \psi^\ell = D^j T^{\mathbf{k}} S\psi^\ell = (S\psi^\ell)_{j, \mathbf{k}}.$$

(b) From Lemma 2.4 we know that  $\tilde{g}(\cdot, \ell, j, \mathbf{k}) = (S^{-1})^* \tilde{\psi}_{j, \mathbf{k}}^\ell$  for  $\ell = 1, \dots, m, j \in \mathbb{Z}, \mathbf{k} \in \mathbb{Z}^n$ . If  $i$  is an integer,  $1 \leq i \leq n$ , then  $T_{x_i} S = ST_{x_i}$  implies that  $(T_{x_i} S)^{-1} = (ST_{x_i})^{-1}$ , i. e.,  $S^{-1} T_{x_i}^{-1} = T_{x_i}^{-1} S^{-1}$ , whence  $(S^{-1} T_{x_i}^{-1})^* = (T_{x_i}^{-1} S^{-1})^*$ . However  $T_{x_i}$  is unitary; thus

$$(S^{-1} T_{x_i}^{-1})^* = (T_{x_i}^{-1})^* (S^{-1})^* = (T_{x_i}^*)^* (S^{-1})^* = T_{x_i} (S^{-1})^*$$

and

$$(T_{x_i}^{-1} S^{-1})^* = (S^{-1})^* (T_{x_i}^{-1})^* = (S^{-1})^* (T_{x_i}^*)^* = (S^{-1})^* T_{x_i}.$$

This implies that  $T_{x_i}(S^{-1})^* = (S^{-1})^*T_{x_i}$ . Since also  $D$  is unitary, we similarly conclude that  $D(S^{-1})^* = (S^{-1})^*D$ . Thus, applying Lemma 2.4 we see that

$$\tilde{g}(\cdot, \ell, j, \mathbf{k}) = (S^{-1})^* \tilde{\psi}_{j,\mathbf{k}}^\ell = (S^{-1})^* D^j T^{\mathbf{k}} \tilde{\psi}^\ell = D^j T^{\mathbf{k}} (S^{-1})^* \tilde{\psi}^\ell = \left( (S^{-1})^* \tilde{\psi}^\ell \right)_{j,\mathbf{k}}.$$

□

### 3. INTRODUCING A MULTIREOLUTION ANALYSIS

A *multiresolution analysis* (MRA) is an ordered pair  $(\{V_j; j \in \mathbb{Z}\}, \varphi)$  where  $\{V_j; j \in \mathbb{Z}\}$  is a sequence of closed linear subspaces of  $L^2(\mathbb{R}^n)$  and  $\varphi \in L^2(\mathbb{R}^n)$ , such that:

$$(3.1) \quad V_j \subset V_{j+1} \text{ for every } j \in \mathbb{Z}$$

$$(3.2) \quad \bigcup_{j \in \mathbb{Z}} V_j \text{ is dense in } L^2(\mathbb{R}^n)$$

$$(3.3) \quad \bigcap_{j \in \mathbb{Z}} V_j = \{0\}$$

$$(3.4) \quad \text{For every } \mathbf{k} \in \mathbb{Z}^n, f(t) \in V_0 \text{ if and only if } f(t - \mathbf{k}) \in V_0$$

$$(3.5) \quad \text{For every } j \in \mathbb{Z}, f(t) \in V_j \text{ if and only if } f(2t) \in V_{j+1}$$

$$(3.6) \quad \{\varphi(\mathbf{x} - \mathbf{k}); \mathbf{k} \in \mathbb{Z}^n\} \text{ is an orthogonal basis of } V_0$$

When we say that a system  $\Psi$  is generated by a multiresolution analysis we mean that there exists a multiresolution analysis  $(\{V_j; j \in \mathbb{Z}\}, \varphi)$  and a sequence  $\{c_{\mathbf{k}}^\ell; \ell = 1, \dots, m, \mathbf{k} \in \mathbb{Z}^n\} \in \ell^2$ , such that

$$(3.7) \quad \psi^\ell(\mathbf{x}) = \sum_{\mathbf{k} \in \mathbb{Z}^n} c_{\mathbf{k}}^\ell \varphi(2\mathbf{x} - \mathbf{k}), \quad \ell = 1, \dots, m$$

in  $L^2(\mathbb{R}^n)$ .

Not every orthonormal or Riesz basis is generated by a multiresolution analysis: Necessary and sufficient conditions for this to occur in the univariate case are discussed in, e. g., [11, Chapter 7] and [18].

If  $\Psi$  is an orthonormal basis of  $L^2(\mathbb{R}^n)$  then  $m = 2^n - 1$  for some  $n \in \mathbb{Z}^+$  (cf. [16, Corollary 1]).

Given a function  $f \in L(\mathbb{R}^n)$ , we define its Fourier transform by

$$\mathcal{F}[f](\mathbf{x}) := \hat{f}(\mathbf{x}) := \int e^{-i\langle \mathbf{x}, t \rangle} f(t) dt.$$

We will also write  $\hat{\psi}_{j,\mathbf{k}}^\ell$  instead of  $\mathcal{F}[\psi_{j,\mathbf{k}}^\ell]$ ,  $\hat{g}_{j,\mathbf{k}}^\ell$  instead of  $\mathcal{F}[g_{j,\mathbf{k}}^\ell]$ , etc.

Equations (3.5) and (3.6) imply that there is a function  $m_0 \in L^2(-\pi, \pi)^n$ ,  $2\pi$ -periodic, such that

$$(3.8) \quad \hat{\varphi}(\mathbf{x}) = m_0(\mathbf{x}/2) \hat{\varphi}(\mathbf{x}/2) \text{ a. e.}$$

In the frequency domain, (3.7) can be written in the form

$$(3.9) \quad \hat{\psi}^\ell(\mathbf{x}) = m_\ell(\mathbf{x}/2) \hat{\varphi}(\mathbf{x}/2) \text{ a. e., } 1 \leq \ell \leq m,$$

where the functions  $m_\ell$  are in  $L^2(-\pi, \pi)^n$  and are  $2\pi$ -periodic.

The function  $\varphi$  is called the *scaling function*. The functions  $\psi^\ell$  are called *mother wavelets*.

Given a multiresolution analysis  $(\{V_j; j \in \mathbb{Z}\}, \varphi)$ ,  $W_j$  will denote the orthogonal complement of  $V_j$  in  $V_{j+1}$ ; thus  $V_{j+1} = V_j \oplus W_j$ .

A basis  $\Psi$  is called *semiorthogonal* if  $j_1 \neq j_2$  implies that  $\langle \psi_{j_1, \mathbf{k}_1}^{\ell_1}, \psi_{j_2, \mathbf{k}_2}^{\ell_2} \rangle = 0$ .

**Lemma 3.1.** *Let  $\mathbb{G}$  be a Riesz basis for  $L^2(\mathbb{R}^n)$ , let  $\Psi$  be a semiorthogonal Riesz basis generated by a multiresolution analysis  $(\{V_j; j \in \mathbb{Z}\}, \varphi)$ , and let  $S$  be the canonical operator from  $\Psi$  onto  $\mathbb{G}$ . Then the operators  $T_{x_1}, \dots, T_{x_n}$  commute with  $S$  on  $V_0^\perp$ .*

*Proof.* Let  $\delta_i := (\delta_{i,1}, \dots, \delta_{i,n})$ , where  $\delta_{i,j}$  is Kröner's delta, and let  $\mathbf{k}(i, j) := 2^j \delta_i + \mathbf{k}$ . If  $j \in \mathbb{Z}_0$ , then  $\mathbf{k}(i, j) \in \mathbb{Z}^n$ , and we have:

$$ST_{x_i} \psi_{j, \mathbf{k}}^\ell = S \psi_{j, \mathbf{k}(i, j)}^\ell = g_{j, \mathbf{k}(i, j)}^\ell = T_{x_i} g_{j, \mathbf{k}}^\ell = T_{x_i} S \psi_{j, \mathbf{k}}^\ell.$$

Since  $\{\psi_{j, \mathbf{k}}^\ell, \ell = 1, \dots, m, j \in \mathbb{Z}_0, \mathbf{k} \in \mathbb{Z}^n\}$  is a Riesz basis for  $V_0^\perp$ , the assertion follows.  $\square$

**Theorem 3.2.** *Let  $\Psi$  be a semiorthogonal Riesz basis of  $L^2(\mathbb{R}^n)$  generated by an MRA  $(\{V_j; j \in \mathbb{Z}\}, \varphi)$ , let  $\mathbb{G}$  be a Riesz basis of  $L^2(\mathbb{R}^n)$ , let  $K$  be the canonical operator from  $\Psi$  onto  $\mathbb{G}$ ,  $\eta := K\varphi$ ,  $U_k := K(V_k)$ ,*

$$r(\mathbf{x}) := \left( \sum_{\mathbf{k} \in \mathbb{Z}^n} |\hat{\eta}(\mathbf{x} + 2\pi\mathbf{k})|^2 \right)^{1/2},$$

*and  $\hat{\phi}(\mathbf{x}) := \hat{\eta}(\mathbf{x})/r(\mathbf{x})$ . Assume that  $K$  commutes with  $T_{x_i}, i = 1, \dots, n$ . Then  $(\{U_j; j \in \mathbb{Z}\}, \phi)$  is an MRA, and  $\mathbb{G}$  is generated by this MRA.*

*Proof.* (a) The continuity of  $K$  implies that  $K(V_k)$  is a closed subspace of  $L^2(\mathbb{R}^n)$ . Moreover, (3.1) trivially implies that

$$K(V_j) \subset K(V_{j+1}) \text{ for every } j \in \mathbb{Z}.$$

(b) Since by hypothesis  $\bigcup_{j \in \mathbb{Z}} V_j$  is dense in  $L^2(\mathbb{R}^n)$  and  $K$  is continuous and its range is  $L^2(\mathbb{R}^n)$ , we readily deduce that also  $\bigcup_{j \in \mathbb{Z}} K(V_j)$  is dense in  $L^2(\mathbb{R}^n)$ .

(c)

$$\bigcap_{j \in \mathbb{Z}} K(V_j) = K \left( \bigcap_{j \in \mathbb{Z}} V_j \right) = K(\{0\}) = \{0\}.$$

(d) Let  $f \in K(V_0)$ . Then  $f = Kg$  for some  $g \in V_0$ . Assume that  $\mathbf{k} \in \mathbb{Z}^n$ . Since the hypotheses imply that  $K$  and  $T^{\mathbf{k}}$  commute, we have:

$$f(\mathbf{x} - \mathbf{k}) = (T^{\mathbf{k}}f)(\mathbf{x}) = (T^{\mathbf{k}}Kg)(\mathbf{x}) = (KT^{\mathbf{k}}g)(\mathbf{x}) \in K(V_0).$$

(e) Let  $f \in K(V_j)$ . Then  $f = Kg$  for some  $g \in V_j$ . Applying Lemma 2.3 we have:

$$f(2\mathbf{x}) = 2^{-n/2}(Df)(\mathbf{x}) = 2^{-n/2}(DKg)(\mathbf{x}) = 2^{-n/2}(KDg)(\mathbf{x}) \in K(V_{j+1}).$$

A similar argument shows that  $f(\mathbf{x}/2) \in K(V_{j-1})$ .

(f) Let  $T$  denote the restriction of  $K$  to  $V_0$ . Then  $T$  is a continuous invertible operator from  $V_0$  to  $K(V_0)$ . Since  $\{\varphi(\mathbf{x} - \mathbf{k}); \mathbf{k} \in \mathbb{Z}^n\}$  is an orthogonal basis of  $V_0$  by hypothesis, and Theorem 2.5 implies that  $\eta_{0, \mathbf{k}} = (T\varphi)_{0, \mathbf{k}}$ , we deduce that  $\{\eta(\mathbf{x} - \mathbf{k}); \mathbf{k} \in \mathbb{Z}^n\}$  is a Riesz basis of  $K(V_0)$ , and from, e. g., [15, p. 26, Theorem 1], we conclude that  $\{\phi(\mathbf{x} - \mathbf{k}); \mathbf{k} \in \mathbb{Z}^n\}$  is an orthonormal basis of  $K(V_0)$ .

We have therefore established that  $(\{U_j; j \in \mathbb{Z}\}, \phi)$  is an MRA.

(g) We now show that  $\mathbb{G}$  is generated by this MRA. The hypotheses imply that (2.1) is satisfied. This can be written in the form

$$\psi^\ell = \sum_{\mathbf{k} \in \mathbb{Z}^n} c_{\mathbf{k}}^\ell DT^{\mathbf{k}} \varphi, \quad 1 \leq \ell \leq m.$$

From Lemma 2.3 we know that  $D$  and  $K$  commute, and from the hypotheses we also know that  $K$  and the  $T_{x_i}$  commute; thus, applying  $K$  to both sides of (3.7) we have:

$$g^\ell = K\psi^\ell = \sum_{\mathbf{k} \in \mathbb{Z}^n} c_{\mathbf{k}}^\ell KDT^{\mathbf{k}} \varphi = \sum_{\mathbf{k} \in \mathbb{Z}^n} c_{\mathbf{k}}^\ell DT^{\mathbf{k}}(K\varphi) = \sum_{\mathbf{k} \in \mathbb{Z}^n} c_{\mathbf{k}}^\ell DT^{\mathbf{k}} \eta, \quad \ell = 1, \dots, m,$$

i. e.,

$$g^\ell(x) = \sum_{\mathbf{k} \in \mathbb{Z}^n} c_{\mathbf{k}}^\ell 2^{n/2} \eta(2x - \mathbf{k}), \quad \ell = 1, \dots, m.$$

Taking the Fourier transform we see that there are functions  $m^\ell(\mathbf{x}) \in L^2(-\pi, \pi)^n$  and  $2\pi$ -periodic, such that

$$\widehat{g}^\ell(\mathbf{x}) = m_\ell(\mathbf{x}/2) \widehat{\eta}(\mathbf{x}/2) = q^\ell(\mathbf{x}/2) \widehat{\phi}(\mathbf{x}/2) \quad \text{a. e.}, \quad \ell = 1, \dots, m.$$

where  $q^\ell(\mathbf{x}) := m_\ell(\mathbf{x})r(\mathbf{x})$ . Since  $r(\mathbf{x})$  is  $2\pi$ -periodic, also the functions  $q^\ell(\mathbf{x})$  are  $2\pi$ -periodic. Moreover, from [15, p. 26, Theorem 1] or [5, Lemma 4.1] we know that  $r(\mathbf{x})$  is bounded. Thus  $q^\ell \in L^2(-\pi, \pi)^n$ ,  $1 \leq \ell \leq n$ , and the assertion follows.  $\square$

We will need the following multivariate version of a Littlewood–Paley identity of Chui and Shi:

**Proposition 3.3.** *Let  $\Psi$  be a frame in  $L^2(\mathbb{R}^n)$  with bounds  $A$  and  $B$ . Then*

$$A \leq \sum_{\ell=1}^m \sum_{j \in \mathbb{Z}} |\widehat{\psi}^\ell(2^{-j}\mathbf{x})|^2 \leq B \quad \text{a. e.}$$

Since the proof is similar to that of [4, Theorem 1] or [5, Theorem 2.1], it will be omitted.

We will call a function *essentially constant* if it equals a constant almost everywhere in its domain. A function  $u(\mathbf{x})$  defined in  $\mathbb{R}^n$  and satisfying  $u(2\mathbf{x}) = u(\mathbf{x})$  a. e. will be called a *2-dilation periodic function*. In a one-variable setting such functions have been used by Papadakis Stavrououlos and Kalouptsidis to study equivalence relations between multiresolution analyses ([14]), and by Dai and Larson in their investigation of the topological and structural properties of the set of all complete wandering vectors for the system  $(D, T)$  acting on  $L^2(R)$  ([7]).

**Proposition 3.4.** *Let  $\Psi$  be a Riesz basis of  $L^2(\mathbb{R}^n)$  with bounds  $A$  and  $B$ , generated by a multiresolution analysis  $(\{V_j; j \in \mathbb{Z}\}, \varphi)$  such that  $\widehat{\varphi}(\mathbf{x}) \neq 0$  a. e.; let  $\eta \in L^2(\mathbb{R}^n)$ ; let  $0 < C < \infty$  and  $|\alpha_\ell|^2 \geq \lambda > 0$ , where  $\alpha_\ell \in \mathbb{C}$  for  $1 \leq \ell \leq m$ . Then the following propositions are equivalent:*

(i) *There is a measurable function  $u$  defined on  $\mathbb{R}^n$ , such that*

- (a)  $\widehat{\eta}(\mathbf{x}) = u(\mathbf{x}) \widehat{\varphi}(\mathbf{x})$  a. e.
- (b)  $u(\mathbf{x})$  is a 2-dilation periodic function.
- (c)  $|u(\mathbf{x})| \leq \sqrt{C/(\lambda B)}$  a. e.
- (d)  $u(\mathbf{x})$  is not essentially constant.

(ii) *The following conditions are satisfied:*

- (1)  $\widehat{\eta}(2\mathbf{x}) = m_0(\mathbf{x})\widehat{\eta}(\mathbf{x})$  a. e.  
 (2)  $\eta \notin V_0$ .  
 (3) If  $\widehat{h}^\ell(\mathbf{x}) := \alpha_\ell m_\ell(\mathbf{x}/2)\widehat{\eta}(\mathbf{x}/2)$ ,  $\ell = 1, \dots, m$ , then  $\mathbb{H}$  is a Bessel sequence in  $L^2(\mathbb{R}^n)$ , with bound  $C$ .

*Proof.* Assume first that (ii) is satisfied.

(a) Let

$$u(\mathbf{x}) := \begin{cases} \widehat{\eta}(\mathbf{x})/\widehat{\varphi}(\mathbf{x}), & \widehat{\varphi}(\mathbf{x}) \neq 0 \\ 0, & \widehat{\varphi}(\mathbf{x}) = 0. \end{cases}$$

(b) If  $\widehat{\varphi}(2\mathbf{x}) \neq 0$  then, from (1) and (3.8),

$$u(2\mathbf{x}) = \frac{\widehat{\eta}(2\mathbf{x})}{\widehat{\varphi}(2\mathbf{x})} = \frac{m_0(\mathbf{x})\widehat{\eta}(\mathbf{x})}{m_0(\mathbf{x})\widehat{\varphi}(\mathbf{x})} = u(\mathbf{x}).$$

Since  $\widehat{\varphi}(\mathbf{x}) \neq 0$  a. e., the assertion follows.

(c) It is clear from (b) that  $u(2^j\mathbf{x}) = u(\mathbf{x})$  a. e. for every  $j \in \mathbb{Z}$ . Since  $\widehat{h}^\ell(2\mathbf{x}) = \alpha_\ell m_\ell(\mathbf{x})\widehat{\eta}(\mathbf{x})$ , we deduce from (1) that, for  $\ell = 1, \dots, m$ ,

$$\widehat{h}^\ell(2^{-j}\mathbf{x}) = \alpha_\ell m_\ell(2^{-j-1}\mathbf{x})u(2^{-j-1}\mathbf{x})\widehat{\varphi}(2^{-j-1}\mathbf{x}) = \alpha_\ell u(\mathbf{x})\widehat{\psi}^\ell(2^{-j}\mathbf{x}) \quad \text{a. e.}$$

Thus, (3) implies that

$$\begin{aligned} C &\geq \sum_{\ell=1}^m \sum_{j \in \mathbb{Z}} |\alpha_\ell \widehat{h}^\ell(2^{-j}\mathbf{x})|^2 = |u(\mathbf{x})|^2 \sum_{\ell=1}^m \sum_{j \in \mathbb{Z}} |\alpha_\ell \widehat{\psi}^\ell(2^{-j}\mathbf{x})|^2 \geq \\ &|u(\mathbf{x})|^2 \lambda \sum_{\ell=1}^m \sum_{j \in \mathbb{Z}} |\widehat{\psi}^\ell(2^{-j}\mathbf{x})|^2 \quad \text{a. e.} \end{aligned}$$

But from Proposition 3.3 we know that

$$\sum_{\ell=1}^m \sum_{j \in \mathbb{Z}} |\widehat{\psi}^\ell(2^{-j}\mathbf{x})|^2 \geq A \quad \text{a. e.},$$

and the assertion follows.

(d) If  $u(\mathbf{x})$  were constant a. e. then  $\widehat{\eta}(\mathbf{x})/\widehat{\varphi}(\mathbf{x}) = k$  a. e., and therefore  $\widehat{\eta}(\mathbf{x}) = k\widehat{\varphi}(\mathbf{x})$  a. e. Thus  $\eta(\mathbf{x}) = k\varphi(\mathbf{x})$  a. e. This implies that  $\eta \in V_0$ , which is a contradiction.

Assume now that (ii) is satisfied.

(1) Clearly  $\widehat{\eta}(2\mathbf{x}) = u(2\mathbf{x})\widehat{\varphi}(2\mathbf{x}) = u(\mathbf{x})m_0(\mathbf{x})\widehat{\varphi}(\mathbf{x}) = m_0(\mathbf{x})\widehat{\eta}(\mathbf{x})$  a. e.

(2) If  $\eta \in V_0$  then, from [5, Proposition 3.1] or [10, Theorem 2.1],  $\eta = \alpha\varphi$  for some  $\alpha \in \mathbb{C}$ , so that  $\widehat{\eta}(\mathbf{x}) = \alpha\widehat{\varphi}(\mathbf{x})$ . Hence  $u(\mathbf{x})\widehat{\varphi}(\mathbf{x}) = \alpha\widehat{\varphi}(\mathbf{x})$ , and therefore  $u(\mathbf{x}) = \alpha$  a.e. on  $\mathbb{R}^n$ , which contradicts (d).

(3)  $\widehat{h}^\ell(\mathbf{x}) = \alpha_\ell m_\ell(\mathbf{x}/2)\widehat{\eta}(\mathbf{x}/2) = \alpha_\ell m_\ell(\mathbf{x}/2)u(\mathbf{x}/2)\widehat{\varphi}(\mathbf{x}/2) = \alpha_\ell u(\mathbf{x})\widehat{\psi}^\ell(\mathbf{x})$  a. e., for  $\ell = 1, 2, \dots, m$ .

If  $f \in L^2(\mathbb{R}^n)$ , a straightforward computation shows that

$$\widehat{f}_{j,k}(\mathbf{x}) = 2^{-jn/2} e^{-i2^j \langle \mathbf{x}, \mathbf{k} \rangle} \widehat{f}(2^{-j}\mathbf{x})$$

(remember that  $\widehat{f}_{j,k}$  stands for  $\mathcal{F}[f_{j,k}]$ ), and we readily conclude that  $\widehat{h}_{j,k}^\ell(\mathbf{x}) = \alpha_\ell u(\mathbf{x})\widehat{\psi}_{j,k}^\ell(\mathbf{x})$  a. e.,  $j \in \mathbb{Z}$ ,  $k \in \mathbb{Z}^n$ ,  $\ell = 1, \dots, m$ . Let  $f \in L^2(\mathbb{R}^n)$  and  $\widehat{g}(\mathbf{x}) :=$

$\widehat{f}(\mathbf{x})\overline{u(\mathbf{x})}$ . Since  $|\widehat{g}(\mathbf{x})|^2 = |\widehat{f}(\mathbf{x})|^2 |u(\mathbf{x})|^2 \leq B|\widehat{f}(\mathbf{x})|^2$ , it is clear that  $\widehat{g}(\mathbf{x}) \in L^2(\mathbb{R}^n)$ . Applying Plancherel's theorem we have:

$$\begin{aligned} \sum_{\ell=1}^m \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^n} |\langle f, h_{j,k}^\ell \rangle|^2 &= (2\pi)^{-2n} \sum_{\ell=1}^m \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^n} |\langle \widehat{f}, \widehat{h}_{j,k}^\ell \rangle|^2 \\ &= (2\pi)^{-2n} \sum_{\ell=1}^m \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^n} |\alpha_\ell|^2 |\langle \widehat{f}, u \widehat{\psi}_{j,k}^\ell \rangle|^2 = (2\pi)^{-2n} \sum_{\ell=1}^m \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^n} |\alpha_\ell|^2 |\langle \widehat{g}, \widehat{\psi}_{j,k}^\ell \rangle|^2 \\ &\leq B\lambda \|g\|^2. \end{aligned}$$

But

$$\|f\|^2 \leq \|g\|^2 \leq \frac{D}{B\lambda} \|f\|^2,$$

whence the assertion follows.  $\square$

Conditions (d) and (2) in the preceding proposition are equivalent and they have been introduced to avoid the trivial case: if  $\eta \in V_0$ , then  $\eta = \alpha\varphi$ ; hence  $K\varphi = \alpha\varphi$ , and we readily conclude that  $K = \alpha I$ , and therefore that  $S = I + K = (1 + \alpha)I$ .

It is easy to find functions that satisfy the conditions (b) (c) and (d) of Proposition 3.4. This follows from part (a) of the following proposition, which is essentially a multivariate version of [14, Proposition 2.5] or the algorithm in [7, Remark 3.6], modulo a small improvement.

**Proposition 3.5.** *Let  $\|\cdot\|$  be any norm in  $\mathbb{R}^n$ , let  $I := \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = 0, \text{ or } 1 \leq \|\mathbf{x}\| < 2\}$ , and let  $u(\mathbf{x})$  be a function defined on  $\mathbb{R}^n$ . Then*

*(a)  $u(\mathbf{x})$  is a 2-dilation periodic function if and only if there is a function  $v(\mathbf{x})$ , defined on  $I$ , such that, if*

$$(3.10) \quad w(\mathbf{x}) := \begin{cases} v(0), & \mathbf{x} = 0 \\ v(2^{-j}\mathbf{x}), & 2^j \leq \|\mathbf{x}\| < 2^{j+1}, j \in \mathbb{Z}, \end{cases}$$

*then  $u(\mathbf{x}) = w(\mathbf{x})$  a. e.*

*(b) If  $u(2\mathbf{x}) = u(\mathbf{x})$  for  $\mathbf{x} \in \mathbb{R}^n$  and  $u(\mathbf{x})$  is  $2\pi$ -periodic, then  $u(\mathbf{x})$  is constant if and only if it is continuous at  $\mathbf{x} = (\pi/2)\mathbf{1}$ .*

*Proof.* (a) Assume first that there is a measurable function  $v(\mathbf{x})$ , defined on  $I$ , such that if  $w(\mathbf{x})$  is defined by (3.10), then  $u(\mathbf{x}) = w(\mathbf{x})$  a. e. If  $\mathbf{x} \neq 0$  then there is a  $j \in \mathbb{Z}$  such that  $2^j \leq \|\mathbf{x}\| < 2^{j+1}$ . Thus  $2^{j+1} \leq \|2\mathbf{x}\| < 2^{j+2}$ , and we have:

$$w(2\mathbf{x}) = v(2^{-(j+1)}2\mathbf{x}) = v(2^{-j}\mathbf{x}) = w(\mathbf{x}) \quad \text{a. e.,}$$

and therefore  $u(2\mathbf{x}) = u(\mathbf{x})$  a. e.

The converse follows by setting  $v(\mathbf{x})$  to be the restriction of  $u(\mathbf{x})$  to  $I$ : Let  $\mathbf{x} \neq 0$ ; then  $2^j \leq \|\mathbf{x}\| < 2^{j+1}$  for some  $j \in \mathbb{Z}$ , and therefore  $u(\mathbf{x}) = u(2^{-j}\mathbf{x}) = v(2^{-j}\mathbf{x})$  a. e.

(b) If  $u(\mathbf{x})$  is a constant, it is clearly continuous at  $\mathbf{x} = (\pi/2)\mathbf{1}$ . To prove the converse note that

$$u(\mathbf{x}) = u(2^{j+2}\mathbf{x}) = u(2^{j+2}\mathbf{x} + (2\pi)\mathbf{1}) = u(2^j\mathbf{x} + (\pi/2)\mathbf{1}),$$

and pass to the limit as  $j \rightarrow -\infty$ .  $\square$

Note that  $(\pi/2)\mathbf{1} \in I$ .

**Proposition 3.6.** *Let  $\Psi$  be a semiorthogonal Riesz basis of  $L^2(\mathbb{R}^n)$  generated by a multiresolution analysis  $(\{V_j; j \in \mathbb{Z}\}, \varphi)$ , let  $\mathbb{G}$  be a Riesz basis of  $L^2(\mathbb{R}^n)$ , let  $\eta \in L^2(\mathbb{R}^n)$  be such that  $\hat{\eta}(\mathbf{x}) = u(\mathbf{x})\hat{\varphi}(\mathbf{x})$  a. e., where  $u(\mathbf{x})$  is a 2-dilation periodic function, and assume that  $\hat{g}^\ell(\mathbf{x}) = m_\ell(\mathbf{x}/2)\hat{\eta}(\mathbf{x}/2)$ ,  $\ell = 1, \dots, m$  a. e. Let  $K$  be a linear operator with domain and range in  $L^2(\mathbb{R}^n)$ , and such that*

$$K\varphi_{0,\mathbf{k}} := \eta_{0,\mathbf{k}}, \quad \mathbf{k} \in \mathbb{Z}^n, \quad K\psi_{j,\mathbf{k}}^\ell := g_{j,\mathbf{k}}^\ell, \quad j \in \mathbb{Z}^+, \quad \mathbf{k} \in \mathbb{Z}^n, \quad \ell = 1, \dots, m.$$

*Then  $K$  is the canonical operator from  $\Psi$  to  $\mathbb{G}$ , and  $K$  commutes with  $T_{x_i}$ ,  $i = 1, \dots, n$ , on  $L^2(\mathbb{R}^n)$ .*

*Proof.* (a) Since  $L^2(\mathbb{R}^n) = V_0 \oplus V_0^\perp$ , the sequence  $\{\varphi_{0,\mathbf{k}}; \mathbf{k} \in \mathbb{Z}^n\} \cup \{\psi_{j,\mathbf{k}}^\ell; \ell = 1, \dots, m, j \in \mathbb{Z}_0, \mathbf{k} \in \mathbb{Z}^n\}$  is a basis for  $L^2(\mathbb{R}^n)$ . Thus, the domain of  $K$  is  $L^2(\mathbb{R}^n)$  and  $K$  is unique.

(b) We claim that  $K\varphi_{-j,0} = \eta_{-j,0}$  for every  $j \in \mathbb{Z}_0$ . This assertion is established by an inductive argument. For  $j = 0$  it is true by hypothesis. To prove the inductive step we proceed as follows: The hypotheses imply that

$$\varphi_{j,\mathbf{k}} = \sum_{\mathbf{r} \in \mathbb{Z}^n} \beta_{\mathbf{r}} \varphi_{j+1,2\mathbf{k}+\mathbf{r}}$$

and

$$\eta_{j,\mathbf{k}} = \sum_{\mathbf{r} \in \mathbb{Z}^n} \beta_{\mathbf{r}} \eta_{j+1,2\mathbf{k}+\mathbf{r}},$$

where  $\{\beta_{\mathbf{r}}; \mathbf{r} \in \mathbb{Z}^n\} \in \ell^2$ . Thus

$$K\varphi_{-j-1,\mathbf{k}} = K \left( \sum_{\mathbf{r} \in \mathbb{Z}^n} \beta_{\mathbf{r}} \varphi_{-j,2\mathbf{k}+\mathbf{r}} \right) = \sum_{\mathbf{r} \in \mathbb{Z}^n} \beta_{\mathbf{r}} \eta_{-j,2\mathbf{k}+\mathbf{r}} = \eta_{-j-1,\mathbf{k}}.$$

(c) We now show that  $K$  is the canonical operator. In view of the hypotheses, it suffices to show that  $K\psi_{j,\mathbf{k}}^\ell = g_{j,\mathbf{k}}^\ell$  for  $\ell = 1, \dots, m, \mathbf{k} \in \mathbb{Z}^n$ , and  $j \leq 0$ .

The hypotheses imply that for any  $j \in \mathbb{Z}$  and any  $\ell$ ,  $1 \leq \ell \leq m$ ,

$$\psi_{j,\mathbf{k}}^\ell = \sum_{\mathbf{r} \in \mathbb{Z}^n} c_{\mathbf{r}}^\ell \varphi_{j+1,2\mathbf{k}+\mathbf{r}},$$

where  $\{c_{\mathbf{r}}^\ell; \mathbf{r} \in \mathbb{Z}^n, \ell = 1, \dots, m\} \in \ell^2$ . If  $j \leq 0$ , then from (b) we have:

$$K\psi_{j,\mathbf{k}}^\ell = K \left( \sum_{\mathbf{r} \in \mathbb{Z}^n} c_{\mathbf{r}}^\ell \varphi_{j+1,2\mathbf{k}+\mathbf{r}} \right) = \sum_{\mathbf{k} \in \mathbb{Z}^n} c_{\mathbf{r}}^\ell \eta_{j+1,2\mathbf{k}+\mathbf{r}} = g_{j,\mathbf{k}}^\ell.$$

(d) From the hypotheses we see that

$$T_{x_i} K\varphi_{0,\mathbf{k}} = T_{x_i} \eta_{0,\mathbf{k}} = \eta_{0,\mathbf{k}_i}$$

and

$$KT_{x_i} \varphi_{0,\mathbf{k}} = K\varphi_{0,\mathbf{k}_i} = \eta_{0,\mathbf{k}_i}.$$

Since  $\{\varphi_{0,\mathbf{k}}; \mathbf{k} \in \mathbb{Z}^n\}$  is a basis for  $V_0$ , we conclude that  $K$  and the  $T_{x_i}$  commute on  $V_0$ . Since  $K$  is the canonical operator from  $\Psi$  to  $\mathbb{G}$ , and  $\mathbb{G}$  is a Riesz basis, the assertion follows from Lemma 3.1.  $\square$

Since  $u(\mathbf{x}/2) = u(\mathbf{x})$  a. e., we have:

**Corollary 3.7.** *Let  $\Psi$  be a semiorthogonal Riesz basis of  $L^2(\mathbb{R}^n)$  generated by a multiresolution analysis  $(\{V_j; j \in \mathbb{Z}\}, \varphi)$ , let  $\mathbb{G}$  be a Riesz basis of  $L^2(\mathbb{R}^n)$ , let  $u(\mathbf{x})$  be a 2-dilation periodic function, and assume that  $\widehat{g}^\ell(\mathbf{x}) = u(\mathbf{x})\widehat{\psi}^\ell(\mathbf{x}/2)$  a. e.,  $\ell = 1, \dots, m$ . Then the canonical operator from  $\Psi$  to  $\mathbb{G}$  commutes with  $T_{x_i}$ ,  $i = 1, \dots, n$ , on  $L^2(\mathbb{R}^n)$ .*

Following, e. g., [7, 12], for an operator  $A : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$  we will use  $\widehat{A}$  or  $\mathcal{F}[A]$  to denote  $\mathcal{F}A\mathcal{F}^{-1}$ . We have:

**Lemma 3.8.** *Let  $A : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$  be a bounded linear operator.*

(a) *If  $A$  has a bounded inverse, then  $(A^*)^{-1} = (A^{-1})^*$ .*

(b)  *$(\mathcal{F}A)^* = \mathcal{F}[A^*]$ .*

*Proof.* Let  $f, g \in L^2(\mathbb{R}^n)$ ,  $f_1 = \mathcal{F}^{-1}f$ , and  $g_1 = \mathcal{F}^{-1}g$ .

(a)

$$\langle (A^{-1})^* A^* f, g \rangle = \langle A^* f, A^{-1} g \rangle = \langle f, g \rangle.$$

Since  $f$  and  $g$  are arbitrary, we conclude that  $(A^{-1})^* A^* = I$ .

(b) Applying Plancherel's theorem we have:

$$\begin{aligned} \langle (\mathcal{F}[A])^* f, g \rangle &= \langle f, \widehat{A}g \rangle = \langle \widehat{f}_1, \mathcal{F}[Ag_1] \rangle = (2\pi)^n \langle f_1, Ag_1 \rangle = (2\pi)^n \langle A^* f_1, g_1 \rangle = \\ &= \langle \mathcal{F}[A^* f_1], \widehat{g}_1 \rangle = \langle \mathcal{F}[A^*]f, g \rangle. \end{aligned}$$

□

**Theorem 3.9.** *Let  $\Psi$  be a semiorthogonal Riesz basis of  $L^2(\mathbb{R}^n)$  with bounds  $A$  and  $B$ , generated by a multiresolution analysis  $(\{U_j; j \in \mathbb{Z}\}, \varphi)$  such that  $\widehat{\varphi}(\mathbf{x}) \neq 0$  a. e. Assume, moreover, that  $\{\psi^1, \dots, \psi^m\}$  is a  $W$ -family. Let  $0 < \varepsilon < 1$  be arbitrary but constant, and let  $u(\mathbf{x})$  be a 2-dilation periodic function in  $L^2(\mathbb{R}^n)$ , not essentially constant, and such that  $|u(\mathbf{x})| \leq \sqrt{\varepsilon A/B}$  a. e. Let  $\widehat{\eta}(\mathbf{x}) := u(\mathbf{x})\widehat{\varphi}(\mathbf{x})$ , and  $\widehat{h}^\ell(\mathbf{x}) := m_\ell(\mathbf{x}/2)\widehat{\eta}(\mathbf{x}/2)$ ,  $\ell = 1, \dots, m$ . If  $K$  is the canonical operator from  $\Psi$  onto  $\mathcal{H}$ ,  $S := I + K$ , and  $g(\cdot, \ell, j, \mathbf{k}) := S\psi_{j,\mathbf{k}}^\ell$ ,  $j \in \mathbb{Z}$ ,  $\mathbf{k} \in \mathbb{Z}^n$ ,  $\ell = 1, \dots, m$ , then*

(a)  *$\{g(\cdot, \ell, j, \mathbf{k}); j \in \mathbb{Z}, \mathbf{k} \in \mathbb{Z}^n, \ell = 1, \dots, m\}$  is a Riesz basis with bounds  $[1 - \sqrt{\varepsilon}]^2 A$  and  $[1 + \sqrt{\varepsilon A/B}]^2 B$ .*

(b) *Let  $\widehat{v}^\ell(\mathbf{x}) := \mathcal{F}[\psi^\ell](\mathbf{x})(1 + u(\mathbf{x}))$ . Then  $g(\cdot, \ell, j, \mathbf{k}) = v_{j,\mathbf{k}}^\ell(\cdot)$ ,  $j \in \mathbb{Z}$ ,  $\mathbf{k} \in \mathbb{Z}^n$ ,  $\ell = 1, \dots, m$ .*

(c) *Let  $\widehat{w}^\ell(\mathbf{x}) := \mathcal{F}[\widetilde{\psi}^\ell](\mathbf{x}) / (1 + \overline{u(\mathbf{x})})$ . Then  $\widetilde{g}(\cdot, \ell, j, \mathbf{k}) = w_{j,\mathbf{k}}^\ell(\cdot)$ ,  $j \in \mathbb{Z}$ ,  $\mathbf{k} \in \mathbb{Z}^n$ ,  $\ell = 1, \dots, m$ .*

(d)  *$\{g(\cdot, \ell, j, \mathbf{k}); j \in \mathbb{Z}, \mathbf{k} \in \mathbb{Z}^n, \ell = 1, \dots, m\}$  is generated by a multiresolution analysis.*

(e) *If the dual sequence  $\widetilde{\Psi}$  is semiorthogonal and is generated by a multiresolution analysis, then also the dual basis  $\widetilde{\mathbb{G}}$  is generated by a multiresolution analysis.*

*Proof.* (a) Proposition 3.4 implies that  $\mathbb{H}$  is a Bessel sequence with bound  $B$ . Since  $B < 1$  and  $\Psi$  is a Riesz basis, the assertion follows from [9, Theorem 5].

(b) Since  $u(2^j \mathbf{x}) = u(\mathbf{x})$  for every  $j \in \mathbb{Z}$  and almost every  $\mathbf{x}$ , we have:

(3.11)

$$\begin{aligned} \widehat{h}_{j,\mathbf{k}}^\ell(\mathbf{x}) &= 2^{-nj/2} e^{-i2^{-j}\langle \mathbf{k}, \mathbf{x} \rangle} \widehat{h}^\ell(2^{-j}\mathbf{x}) = 2^{-nj/2} e^{-i2^{-j}\langle \mathbf{k}, \mathbf{x} \rangle} m_\ell(2^{-j-1}\mathbf{x}) \widehat{\eta}(2^{-j-1}\mathbf{x}) \\ &= 2^{-nj/2} e^{-i2^{-j}\langle \mathbf{k}, \mathbf{x} \rangle} m_\ell(2^{-j-1}\mathbf{x}) u(2^{-j-1}\mathbf{x}) \widehat{\varphi}(2^{-j-1}\mathbf{x}) = u(\mathbf{x}) \widehat{\psi}_{j,\mathbf{k}}^\ell(\mathbf{x}). \end{aligned}$$



But

$$g(\cdot, \ell, j, \mathbf{k}) = \psi_{j,\mathbf{k}}^\ell + K\psi_{j,\mathbf{k}}^\ell = \psi_{j,\mathbf{k}}^\ell + h_{j,\mathbf{k}}^\ell.$$

Thus,

$$\widehat{g}(\mathbf{x}, \ell, j, \mathbf{k}) = \widehat{\psi}_{j,\mathbf{k}}^\ell(\mathbf{x}) + \widehat{h}_{j,\mathbf{k}}^\ell(\mathbf{x}) = \widehat{\psi}_{j,\mathbf{k}}^\ell(\mathbf{x})[1 + u(\mathbf{x})] = \widehat{v}_{j,\mathbf{k}}^\ell(\mathbf{x}) \quad \text{a. e.}$$

(c) Equation (2.1) implies that

$$f = \sum_{\ell=1}^m \sum_{j \in \mathbb{Z}} \sum_{\mathbf{k} \in \mathbb{Z}^n} \langle \check{f}, \widetilde{\psi}_{j,\mathbf{k}}^\ell \rangle \widehat{\psi}_{j,\mathbf{k}}^\ell.$$

From (2.1) we also obtain

$$Kf = \sum_{\ell=1}^m \sum_{j \in \mathbb{Z}} \sum_{\mathbf{k} \in \mathbb{Z}^n} \langle f, \widetilde{\psi}_{j,\mathbf{k}}^\ell \rangle h_{j,\mathbf{k}}^\ell.$$

Applying (3.11) we have:

$$\begin{aligned} (3.12) \quad \widehat{K}f(\mathbf{x}) &= [\mathcal{F}K\mathcal{F}^{-1}]f(\mathbf{x}) = \sum_{\ell=1}^m \sum_{j \in \mathbb{Z}} \sum_{\mathbf{k} \in \mathbb{Z}^n} \langle \mathcal{F}^{-1}f, \widetilde{\psi}_{j,\mathbf{k}}^\ell \rangle \widehat{h}_{j,\mathbf{k}}^\ell \\ &= u(\mathbf{x}) \sum_{\ell=1}^m \sum_{j \in \mathbb{Z}} \sum_{\mathbf{k} \in \mathbb{Z}^n} \langle \mathcal{F}^{-1}f, \widetilde{\psi}_{j,\mathbf{k}}^\ell \rangle \widehat{\psi}_{j,\mathbf{k}}^\ell(\mathbf{x}) = u(\mathbf{x})f(\mathbf{x}) \quad \text{a. e.} \end{aligned}$$

Let  $f, g \in L^2(\mathbb{R}^n)$ . Applying (3.12) we obtain:

$$\langle (\widehat{K})^* f, g \rangle = \langle f, \widehat{K}g \rangle = \langle f, ug \rangle = \langle f\overline{u}, g \rangle,$$

and we conclude that

$$(\widehat{K})^* f(\mathbf{x}) = f(\mathbf{x})\overline{u}(\mathbf{x}) \quad \text{a. e.}$$

An inductive argument shows that

$$[(\widehat{K})^*]^j f(\mathbf{x}) = f(\mathbf{x})[\overline{u(\mathbf{x})}]^j, \quad j \in \mathbb{Z}^+, \quad \text{a. e.}$$

As remarked above, Proposition 3.4 implies that  $\mathbb{H}$  is a Bessel sequence with bound  $\varepsilon A$ . Applying Lemma 2.1 we conclude that  $\|K^*\| = \|K\| = \sqrt{\varepsilon} < 1$ . Thus

$$[(\widehat{S})^*]^{-1} = [I + (\widehat{K})^*]^{-1} = I + \sum_{j \in \mathbb{Z}^+} (-1)^j [(\widehat{K})^*]^j.$$

Hence, since  $|u(\mathbf{x})| < 1$ ,

$$[(\widehat{S})^*]^{-1} f(\mathbf{x}) = f(\mathbf{x}) \left[ 1 + \sum_{j \in \mathbb{Z}^+} (-1)^j [\overline{u(\mathbf{x})}]^j \right] = f(\mathbf{x})/[1 + \overline{u(\mathbf{x})}] \quad \text{a. e.}$$

Therefore, applying Lemma 3.8 we conclude that

$$(3.13) \quad \mathcal{F}[(S^{-1})^* f](\mathbf{x}) = [(\widehat{S})^*]^{-1} \widehat{f}(\mathbf{x}) = \widehat{f}(\mathbf{x})/[1 + \overline{u(\mathbf{x})}].$$

From Corollary 3.7,  $K$  commutes with  $T_{x_i}$ ,  $i = 1, \dots, n$ , on  $L^2(\mathbb{R}^n)$ ; thus *a fortiori*  $S$  commutes with  $T_{x_i}$ ,  $i = 1, \dots, n$ , and the assertion follows from Theorem 2.5.

(d) Since  $S$  commutes with the  $T_{x_i}$ , this is a direct consequence of Theorem 3.2.

(e) Let  $U$  denote the canonical operator from  $\widetilde{\Psi}$  to  $\widetilde{\mathbb{G}}$ . Combining Lemma 2.4 and (3.13) we deduce that

$$\mathcal{F}[Uf](\mathbf{x}) = \widehat{f}(\mathbf{x})/[1 + \overline{u(\mathbf{x})}], \quad f \in L^2(\mathbb{R}^n).$$

Since  $1/[1 + \overline{u(\mathbf{x})}]$  is a 2-dilation periodic function, applying Corollary 3.7 we deduce that  $U$  commutes with the  $T_{x_i}$ , and the assertion follows by another application of Theorem 3.2.  $\square$

For the sake of simplicity, we will now consider the univariate case. We will omit the subscript and superscript  $\ell$ , since they are now redundant. If  $\mathbb{G}$  is a semiorthogonal basis of  $L^2(\mathbb{R})$  then it is generated by a  $W$ -family ([2, Theorem 3.25]). Moreover, combining Theorem 3.9 with [2, Theorem 3.25] we deduce that

$$\widehat{w}(x) = \frac{\widehat{v}(x)}{\sum_{k \in \mathbb{Z}} |\widehat{v}(x + 2k\pi)|^2} \quad \text{a. e.},$$

or

$$\frac{\mathcal{F}[\psi](x)(1 + u(x))}{\sum_{k \in \mathbb{Z}} |\widehat{\psi}(x + 2k\pi)|^2 |1 + u(x + 2k\pi)|^2} = \frac{\mathcal{F}[\widetilde{\psi}](x)}{1 + \overline{u(x)}} \quad \text{a. e.}$$

Thus, we obtain the following:

**Corollary 3.10.** *Let  $n = 1$  and assume that  $\mathbb{G}$  is generated in the manner described in Theorem 3.9. If  $\mathbb{G}$  is semiorthogonal, then*

$$\mathcal{F}[\widetilde{\psi}](x) \sum_{k \in \mathbb{Z}} |\widehat{\psi}(x + 2k\pi)|^2 |1 + u(x + 2k\pi)|^2 = \mathcal{F}[\psi](x) |1 + u(x)|^2 \quad \text{a. e.}$$

It is now easy to find examples of affine Riesz bases that are not semiorthogonal but are generated by  $W$ -families: Let  $v(x)$  be defined on  $(-2, -1] \cup \{0\} \cup [1, 2)$  as follows:

$$v(x) := \begin{cases} 0, & x = 0 \text{ or } 1 \leq |x| < \pi/2 \\ \sqrt{\varepsilon}, & \pi/2 \leq |x| < 2, \end{cases}$$

where  $0 < \varepsilon < 1$  is arbitrary but fixed, let  $w(x)$  be defined as in (3.10), and  $u = w$ . Then  $u(x)$  is not constant, satisfies  $u(2x) = u(x)$  for every real  $x$ , and  $|u(x)| \leq \sqrt{\varepsilon}$ . Let  $\Psi$  be an orthonormal basis of  $L^2(\mathbb{R})$  generated by a MRA  $(\{U_j; j \in \mathbb{Z}\}, \varphi)$  such that  $\varphi(x) \neq 0$  a. e., let  $\mathbb{G}$  be constructed from  $\Psi$  and  $u$  in the manner described in Theorem 3.9, and assume that  $\mathbb{G}$  is semiorthogonal. Since  $u(x)$  is discontinuous at  $x = \pi/2$ , Proposition 3.5 implies that  $u(x)$  is not  $2\pi$ -periodic. In view of [5, Lemma 4.1] there is a set  $J$ , of strictly positive measure, such that  $x \in J$  implies that

$$\sum_{j \in \mathbb{Z}} |\widehat{\psi}(2^{-j}x)|^2 = 1,$$

and either  $u(x) = 0$ ,  $u(x + 2\pi) = \sqrt{\varepsilon}$  for every  $x \in J$ , or  $u(x) = \sqrt{\varepsilon}$ ,  $u(x + 2\pi) = 0$  for every  $x \in J$ . If the first alternative holds, then

$$\begin{aligned} \sum_{k \in \mathbb{Z}} |\widehat{\psi}(x + 2k\pi)|^2 |1 + u(x + 2k\pi)|^2 &\geq (2\sqrt{\varepsilon} + \varepsilon) |\widehat{\psi}(x + 2\pi)|^2 + \sum_{k \in \mathbb{Z}} |\widehat{\psi}(x + 2k\pi)|^2 = \\ &= (2\sqrt{\varepsilon} + \varepsilon) + 1 > 1 = (1 + u(x))^2 \end{aligned}$$

for every  $x \in J$ . On the other hand, if the second alternative holds, then

$$\begin{aligned} \sum_{k \in \mathbb{Z}} |\widehat{\psi}(x + 2k\pi)|^2 |1 + u(x + 2k\pi)|^2 &\geq (2\sqrt{\varepsilon} + \varepsilon) |\widehat{\psi}(x)|^2 + \sum_{k \in \mathbb{Z}} |\widehat{\psi}(x + 2k\pi)|^2 = \\ &= (2\sqrt{\varepsilon} + \varepsilon) + 1 > 1 = (1 + u(x))^2 \end{aligned}$$

The orthonormality of  $\Psi$  implies that  $\tilde{\psi} = \psi$ . Moreover, from (3.9) we deduce that  $\hat{\psi}(x) \neq 0$  a. e. Thus, the subset  $J_1$  of  $J$  of points  $x$  for which  $\hat{\psi}(x) \neq 0$  has strictly positive measure, and we conclude that

$$\mathcal{F}[\tilde{\psi}](x) \sum_{k \in \mathbb{Z}} |\hat{\psi}(x + 2k\pi)|^2 |1 + u(x + 2k\pi)|^2 > \mathcal{F}[\psi](x) |1 + u(x)|^2$$

for every  $x \in J_1$ . We have therefore obtained a contradiction to Corollary 3.10, and we conclude that  $\Psi$  cannot be semiorthogonal.

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# Beyond the Monomiality: the Monumbrality Principle

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## Abstract

The monomiality principle has allowed the derivation, in a fairly straightforward way, of the properties of many families of special polynomials, recognized as quasi monomials. The family of quasi-monomials is large, but not all the special polynomials can be framed within this category. In this paper we will prove the existence of a larger family, including the quasi-monomials, by taking advantage from the umbral calculus and by introducing the concept of monumbrality, which appears very useful to get a deeper understanding in the nature and role of special polynomials.

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## 1. Introduction

According to the monomiality principle [3] a polynomial  $p_n(x)$  is said a quasi-monomial if two operators  $\hat{P}$  and  $\hat{M}$  usually referred as “derivative”

and “multiplicative” operators can be defined in such a way that

$$\begin{aligned}\hat{P}p_n(x) &= np_{n-1}(x) \\ \hat{M}p_n(x) &= p_{n+1}(x).\end{aligned}\tag{1}$$

If  $\hat{P}$  and  $\hat{M}$  have a differential realization, it is readily understood that

$$\hat{M}\hat{P}p_n(x) = np_n(x)\tag{2}$$

provides the differential equation for the polynomial, furthermore setting  $p_0(x) = 1$  the following important property holds

$$\hat{M}^n \{1\} = p_n(x).\tag{3}$$

Examples of quasi-monomials are the Hermite and Laguerre polynomials and the relevant derivative and multiplicative operators are reported below

a) *Hermite Polynomials*

$$p_n(x) \equiv H_n(x, y) = n! \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \frac{y^{n-2r} x^r}{(n-2r)! r!},\tag{4}$$

$$\hat{P} = \frac{\partial}{\partial x}, \quad \hat{M} = x + 2y \frac{\partial}{\partial x},$$

b) *Laguerre Polynomials*

$$p_n(x) \equiv \mathcal{L}_n(x, y) = n! \sum_{r=0}^n \frac{y^{n-r} x^r}{(n-r)! (r!)^2},\tag{5}$$

$$\hat{P} = -\frac{\partial}{\partial x} x \frac{\partial}{\partial x}, \quad \hat{M} = y - \hat{\mathcal{D}}_x^{-1},$$

where  $\hat{\mathcal{D}}_x^{-1}$  denotes the negative derivative operator and is defined in such a way that

$$\hat{\mathcal{D}}_x^{-n} \{1\} = \frac{x^n}{n!}\tag{6}$$

A large body of polynomials has been recognized as quasi-monomials [4] and such a classification has greatly simplified the derivation of their properties. In the following we will extend the concept of quasi monomiality

and will discuss a more general classification, including the monomiality itself and based on the elementary umbral calculus [5]. We will prove the usefulness of this more general point of view and show that Appell [2] and Laguerre type [1] polynomials can be framed within this context.

## 2. The Monumbrality Principle

The symbol  $a$  will be said an “umbral” operator if

$$a^n = a_n \quad (7)$$

and  $a_n$  may denote the  $n^{th}$  term of a sequence, a discrete function, etc.

According to such a definition we can introduce the polynomials

$$p_n(x) = (a + x)^n = \sum_{s=0}^n \binom{n}{s} a_{n-s} x^s, \quad (8)$$

with  $a_n$  independent of  $x$ .

The polynomials (8) are evidently quasi monomial under the action of the operators

$$\begin{aligned} \hat{P} &= \frac{d}{dx}, \\ \hat{M} &= (a + x). \end{aligned} \quad (9)$$

According to the identity (2) the polynomials (8) satisfy the equation

$$(a + x) \frac{d}{dx} p_n(x) = n p_n(x) \quad (10)$$

and since

$$\pi_n(x) = a p'_n(x) = n \sum_{s=0}^{n-1} \binom{n-1}{s} a_{n-s} x^s \quad (11)$$

defines a new family of polynomials not trivially linked to the  $p_n(x)$ , we find from eq. (10) the differential relation

$$\pi_n(x) + x \frac{d}{dx} p_n(x) = n p_n(x). \quad (12)$$

To better clarify the nature of polynomials (8) we note that the relevant generating function can be written as [4]

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{t^n}{n!} p_n(x) &= e^{t\hat{M}} \{1\} = A(t)e^{xt}, \\ A(t) &= \sum_{s=0}^{\infty} \frac{a_s}{s!} t^s, \end{aligned} \quad (13)$$

and therefore they can be identified as belonging to the Appell family [2]. An example of Appell is provided by Bernoulli polynomials for which

$$\begin{aligned} A(t) &= \frac{t}{e^t - 1} = \sum_{n=0}^{\infty} \frac{t^n}{n!} B_n, \\ |t| &< \pi, \end{aligned} \quad (14)$$

with  $B_n$  being the Bernoulli numbers. In this case we have

$$\begin{aligned} p_n(x) &\equiv \sum_{s=0}^n \binom{n}{s} B_{n-s} x^s, \\ \pi_n(x) &= n \sum_{s=0}^{n-1} \binom{n-1}{s} B_{n-s} x^s \end{aligned} \quad (15)$$

and the equation linking the Bernoulli polynomials and their one unity shifted partners is given by (12).

More in general we will say that when a polynomial is a quasi monomial under the action of operators containing umbrals as the operators provided by eq. (9) it will be said a quasi-monumbral and further example will be discussed in the forthcoming section.

### 3. Monumbrals of the Hermite and Laguerre type

Eq. (4) suggests the introduction of the following quasi-monumbrals operators

$$\begin{aligned} \hat{M} &= x + 2a \frac{d}{dx}, \\ \hat{P} &= \frac{d}{dx} \end{aligned} \quad (16)$$



it is easily realized that

$$\left(x + 2a \frac{d}{dx}\right)^n \{1\} = h_n(x, a) = n! \sum_{s=0}^{\left[\frac{n}{2}\right]} \frac{x^{n-2s} a_s}{s!(n-2s)!} \quad (17)$$

and that the polynomial  $h_n(x, a)$  we will call the umbral-Hermite polynomial satisfy the differential relation

$$2 \frac{d^2}{dx^2} \eta_n(x, a) + x \frac{d}{dx} h_n(x, a) = n h_n(x, a), \quad (18)$$

linking  $h_n(x, a)$  to the shifted partner  $\eta_n(x, a)$  polynomials defined as

$$\eta_n(x, a) = n! \sum_{s=0}^{\left[\frac{n}{2}\right]} \frac{x^{n-2s} a_{s+1}}{s!(n-2s)!} . \quad (19)$$

In the case of Laguerre polynomials we define

$$\hat{P} = -\frac{d}{dx} x \frac{d}{dx}, \quad \hat{M} = a - \hat{\mathcal{D}}_x^{-1} \quad (20)$$

and prove that the umbral-Laguerre are

$$l_n(x, a) = (a - \hat{\mathcal{D}}_x^{-1})^n = \sum_{s=0}^n \binom{n}{s} a_{n-s} \frac{x^s}{s!}, \quad (21)$$

with the characteristic differential relation

$$x \frac{d^2}{dx^2} \lambda_n(x, a) + \frac{d}{dx} \lambda_n(x, a) - x \frac{d}{dx} l_n(x, a) + n l_n(x, a) = 0, \quad (22)$$

where

$$\lambda_n(x, a) = \sum_{s=0}^n \binom{n}{s} a_{n-s+1} \frac{x^s}{s!} \quad (23)$$

The examples of this section show the flexibility of the method and further application stating its generality will be presented in the forthcoming section.

#### 4. Concluding remarks

One of the advantages offered by the formalism associated with the monomiality principle is the possibility of using rules of operational nature to derive, in a direct way, results which are hardly obtained with conventional means.

We will show that, the monumbrality principle discussed in this paper, provides a similar and even more flexible formalism.

We first consider the derivation of the generating function of  $h_n(x, a)$  polynomials.

a) we consider the identity

$$e^{t\hat{M}} = e^{t(x+2a\frac{d}{dx})} \quad (24)$$

b) we apply the Weyl decoupling rule [4] thus getting

$$\begin{aligned} e^{t(x+2a\frac{d}{dx})} \{1\} &= e^{tx+t^2a} e^{2t\frac{d}{dx}} \{1\} = e^{tx+t^2a} = \\ &= e^{tx} A(t^2). \end{aligned} \quad (25)$$

Eq. (17) states that when the  $n^{th}$  power of the multiplicative operator associated with the umbral-Hermite acts on unity yields the  $h_n(x, a)$ , on the other side if we expand the operator itself without the assumption of its action on the unity (or on a constant) we find

$$\left(x + 2a\frac{d}{dx}\right)^n = \sum_{s=0}^n \binom{n}{s} h_{n-s}(x, a) \frac{d^s}{dx^s} \quad (26)$$

this relation, providing the umbral counterpart of the Burchnal identity (see [2] and references therein), can be proved directly from eq. (25).

As to the umbral-Laguerre we find

$$\begin{aligned} e^{t\hat{M}} &= e^{t(a-\hat{D}_x^{-1})} \{1\} = B(t)C_0(xt), \\ B(t) &= \sum_{s=0}^{\infty} \frac{b_s}{s!} t^s, \quad C_0(x) = \sum_{r=0}^{\infty} \frac{(-x)^r}{(r!)^2} \end{aligned} \quad (27)$$

thus finding that the  $l_n(x, a)$  belong to the family of polynomials discussed in ref. [5].

Further comments will be presented in a forthcoming investigation where we will discuss a more systematic approach to the problem.

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# Distribution of Eigenvalues in Gaps of the Essential Spectrum of Sturm-Liouville Operators — a Numerical Approach

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## Abstract

This paper reports on a new numerical procedure to count eigenvalues in spectral gaps for a class of perturbed periodic Sturm-Liouville operators. It is motivated by the desire to analyse the distribution of eigenvalues in the dense point spectrum of  $d$ -dimensional radially periodic Schrödinger operators. Our numerical results indicate that the well-known asymptotic formula for the large-coupling limit gives a good description already for moderate values of the coupling constant.

**Keywords:** Sturm-Liouville operators, discrete spectrum, Schrödinger operators, dense point spectrum, numerical methods.

## 0 Introduction

The starting point of this investigation is the observation that spherically symmetric, radially periodic Schrödinger operators possess intervals of dense point spectrum. Indeed, let  $V(x) = q(|x|)$  for  $x \in \mathbb{R}^d$ ,  $d \in \mathbb{N} \setminus \{1\}$ , with a bounded, non-constant periodic function  $q$ ; the corresponding Schrödinger operator  $-\Delta + V$ , defined on  $C_0^\infty(\mathbb{R}^d)$ , has a unique self-adjoint extension  $T$  in  $L_2(\mathbb{R}^d)$ . The essential spectrum of  $T$  is a union of alternating intervals of absolutely continuous and dense point spectrum (cf. [9, Theorem 1 and Corollary 2]). We set out to approach, by numerical calculation, the open questions of how eigenvalues of  $T$  are distributed in the intervals  $I_n = (M_n, \mu_n)$  of dense point spectrum and if in the case  $d = 2$  there are eigenvalues below the essential spectrum. By spherical separation, all these eigenvalues must be eigenvalues of Sturm-Liouville operators on  $(0, \infty)$  of the type  $t_c = -\frac{d^2}{dr^2} + q(r) + \frac{c}{r^2}$ , where  $c$  runs through the values  $c_l := l(l+d-2) + \frac{1}{4}(d-1)(d-3)$ ,  $l \in \mathbb{N}_0$ . Some early numerical experiments were performed using the SLEIGN2 code (cf. [2, 3]). Since our first goal was the computation of eigenvalue approximations in the gaps of the essential spectrum, SLEIGN2 cannot be used in a naïve manner, because this will only reveal eigenvalues *below* the essential spectrum, all other eigenvalues being reported as lying at the base of the essential spectrum. Therefore, with the theory of [1, 17] in mind, we replaced the problem on  $(0, \infty)$  by one on  $[a, b] \subset (0, \infty)$  with Dirichlet boundary conditions. Among the first results was the discovery of one eigenvalue below the essential spectrum of  $T$  in the two-dimensional case (cf. [4]). Further

analysis revealed the existence of infinitely many discrete eigenvalues accumulating at the lowest point of the essential spectrum (cf. [13]).

A similar analytic result holds in the gaps of the essential spectrum of the operators  $t_c$ . Although for fixed  $l$  only finitely many eigenvalues of  $t_{c_l}$  will lie in  $I_n$  for sufficiently large  $n$  (cf. [12, Corollary 3]), there will be infinitely many of them, accumulating at  $M_n$ , for fixed  $n$  and those  $l$  for which  $c_l$  exceeds a critical coupling constant depending on  $n$  (cf. [14]). Therefore, in order to get a more detailed picture of the distribution of the dense eigenvalues of  $T$ , we are forced to consider primarily large values of  $l$ . In this case the angular momentum term is of comparable size to the periodic potential on a correspondingly large interval, which requires computation on an interval  $[a, b]$  of considerable length. This led to serious numerical problems in the performance of SLEIGN2, in addition to the difficulty of justifying the approximation of  $(0, \infty)$  by a compact interval. So we changed paradigms from *calculating* eigenvalues to *counting* them. The theoretical foundation for this method, based on relative oscillation theory, will be developed in Section 1. Its realisation as a computer code and the results of our numerical investigations will be presented in Section 2. They led us to conjecture that the asymptotic formula for the number of eigenvalues in a fixed interval as  $c$  tends to infinity may in fact give a good description for finite values of  $c$ , a question which is discussed in Section 3.

## 1 Fundamental estimates

In this section, we provide estimates for the quality of the approximations used in our calculation of eigenvalue counts in spectral gaps of perturbed periodic Sturm-Liouville operators. We assume the following *general hypotheses* throughout.

Let  $q \in L_{1,\text{loc}}(\mathbb{R})$  be an  $\alpha$ -periodic, real-valued potential,  $\alpha > 0$ , and  $t_0$  the self-adjoint realisation of the Sturm-Liouville differential expression  $-\frac{d^2}{dr^2} + q(r)$  on  $\mathbb{R}$ . Furthermore, let  $\tilde{q} \in L_{1,\text{loc}}((0, \infty))$  be a real-valued perturbation such that  $q(r) + \tilde{q}(r) \rightarrow \infty$  as  $r \rightarrow 0$ ,  $\tilde{q}(r) \rightarrow 0$  as  $r \rightarrow \infty$ , and the perturbed Sturm-Liouville differential expression

$$-\frac{d^2}{dr^2} + q(r) + \tilde{q}(r)$$

is in the limit-point case at 0. Under these hypotheses, the operator is also limit-point at  $\infty$  (cf. [18, Theorem 5.7, in conjunction with Theorems 6.2 and 12.3]), and we denote by  $t$  the corresponding self-adjoint operator on  $(0, \infty)$  (cf. [18, Theorem 5.8]).

It is well known ([5, §5.3], [18, Section 12]) that the spectrum of  $t_0$  has band structure, conveniently described in terms of the discriminant  $D(\lambda)$  (the trace of the canonical fundamental system after one period) of the associated periodic Sturm-Liouville equation

$$-u'' + qu = \lambda u. \tag{1}$$

$D$  is a monotonic function with range  $(-2, 2)$  on each of an infinite sequence of disjoint open intervals (*stability intervals*); the interior of the interval separating the  $n$ th and  $(n+1)$ th stability intervals is called the  $n$ th *instability interval*  $I_n$  and may be empty. The spectrum of  $t_0$  is the closure of the union of all stability intervals, and hence each spectral gap coincides with some (non-empty) instability interval. Sometimes one adds  $I_0 = (-\infty, \inf \sigma(t_0))$ , but we will only consider proper gaps here. In particular, the spectrum of  $t_0$  is purely essential, and under the above general hypotheses the essential spectra of  $t$  and  $t_0$  are the same by virtue of Glazman's decomposition principle [7, Chapter I, Theorem 23]. Moreover, the spectrum of  $t$  is purely absolutely continuous inside the bands of  $t_0$ ; cf. [16, Theorem 2b].

The following proposition gives an estimate of the number of discrete eigenvalues of  $t$  in an interval compactly contained in one of the gaps in its essential spectrum. (In what appears below, *solution* will always mean non-trivial real-valued solution.)

**Proposition 1** *Let  $\lambda_1, \lambda_2 \in I_n$ , with  $\lambda_1 < \lambda_2$ . Choose constants  $a > 0$ ,  $m_1, m_2 \in \mathbb{N}$  such that  $\inf_{r \in (0, a)} \{q(r) + \tilde{q}(r)\} > \lambda_2$ , and  $|\tilde{q}(r)| \leq \text{dist}(\lambda_j, \sigma(t_0))$  for  $r \geq m_j \alpha$  and  $j \in \{1, 2\}$ . Denote by  $N_j$  the number of zeros in  $(a, m_j \alpha]$  of a solution  $u_j$  of*

$$-u'' + (q + \tilde{q})u = \lambda_j u, \quad (2)$$

*satisfying the boundary condition  $u_j(a) = 0$ .*

*Then for the total multiplicity  $\mathcal{N}_t(\lambda_1, \lambda_2)$  of the spectrum of  $t$  in  $(\lambda_1, \lambda_2)$ , the estimate*

$$\mathcal{N}_t(\lambda_1, \lambda_2) - (N_2 - N_1 + (m_1 - m_2)n) \in \{-4, \dots, 3\}$$

*holds.*

The proof of Proposition 1 requires the following preparatory observation.

**Lemma 1** *Let  $\lambda \in I_n$ , and let  $J \subset (0, \infty)$  be a half-open interval of length  $k\alpha$ ,  $k \in \mathbb{N}$ , such that  $|\tilde{q}| \leq \text{dist}(\lambda, \sigma(t_0))$  on  $J$ . Then any solution  $u$  of*

$$-u'' + (q + \tilde{q})u = \lambda u$$

*has at least  $nk - 1$ , and at most  $nk + 1$  zeros in  $J$ .*

*Proof.* Consider  $I_n = (M_n, \mu_n)$ . Then the periodic ( $n$  even) or semi-periodic ( $n$  odd) Floquet solutions of equation (1) with spectral parameter  $M_n$  or  $\mu_n$  have exactly  $n$  zeros in the period interval  $[0, \alpha]$  (cf. [5, Theorem 3.1.2]); hence they have exactly  $nk$  zeros on  $J$ . The assertion now follows by Sturm comparison [8, Section XI.3]. ■

*Remark.* As a consequence of Lemma 1 (with  $\tilde{q} = 0$ ), points of the  $n$ -th instability interval of  $t_0$  are characterised by the following two properties:

- (i)  $|D(\lambda)| > 2$ , where  $D$  is the discriminant of the unperturbed periodic equation (1);
- (ii) if  $u$  is any solution of (1), then  $u$  has between  $3n - 1$  and  $3n + 1$  zeros on  $[0, 3\alpha]$ .

These properties are used in the next section to calculate  $I_n$  automatically to arbitrary precision.

*Proof of Proposition 1.* Let  $m \in \mathbb{N}$ ,  $m > \max\{m_1, m_2\}$ . By Lemma 1, the solution  $u_j$  has  $N_j + (m - m_j)n + \{-1, 0, 1\}$  zeros on  $(a, m\alpha]$ . Let  $v_j$  be a solution of (2) which is square-integrable at 0; such a solution exists, as  $\lambda_j \notin \sigma_e(t)$  (cf. [18, Theorem 11.5c]). By relative oscillation theory ([6, Theorem 1.2a]), we then have  $\mathcal{N}_t(\lambda_1, \lambda_2) = \liminf_{b \rightarrow \infty} N(b)$ , where  $N(b)$  denotes the difference of the number of zeros of  $v_2$  and  $v_1$  on  $(0, b]$ . By Sturm comparison,  $v_j$  has exactly one zero strictly between consecutive zeros of  $u_j$ , and it has at most one zero on  $(0, a]$ , as the equation (2) is disconjugate on this interval. We can even show that  $v_j$  has no zero on  $(0, a]$ . Indeed, assume on the contrary that  $\xi \in (0, a]$  is a zero of  $v_j$ . Then the operator

$$s = -\frac{d^2}{dr^2} + q(r) + \tilde{q}(r) - \lambda_j$$

with domain  $D(s) = \{u \in L_2(0, \xi); u, u' \in AC_{\text{loc}}(0, \xi), su \in L_2(0, \xi), u(\xi) = 0\}$  is self-adjoint, and its restriction  $s_0$  to  $D(s_0) := \{u \in D(s); \text{supp } u \subset (0, \xi]\}$  is essentially

self-adjoint ([11, Satz 1]). As  $v_j \in D(s)$ , there is a sequence  $(w_k)_{k \in \mathbb{N}}$  in  $D(s_0)$  such that

$$\lim_{k \rightarrow \infty} (\|v_j - w_k\| + \|sv_j - sw_k\|) = 0.$$

Hence

$$\begin{aligned} \inf_{(0, \xi)} (q + \tilde{q} - \lambda_j) \|w_k\|^2 &\leq \int_0^\xi (|w'_k|^2 + (q + \tilde{q} - \lambda_j) |w_k|^2) \\ &= \int_0^\xi (-w''_k + (q + \tilde{q} - \lambda_j) w_k) \overline{w_k} \\ &= (sw_k, w_k) \\ &\leq \|sv_j\| \|w_k - v_j\| + \|sw_k - sv_j\| \|w_k\|. \end{aligned}$$

In the limit  $k \rightarrow \infty$ , this implies  $\inf_{(0, \xi)} (q + \tilde{q} - \lambda_j) \leq 0$ , contrary to the choice of  $a$ .

Consequently, the number of zeros of  $v_j$  on  $(0, m\alpha]$  is  $N_j + (m - m_j)n + \{-1, 0, 1, 2\}$  (note that  $v_j$  may have one additional zero after the last zero of  $u_j$ ), and we find

$$N(m\alpha) \in N_2 - N_1 + (m_1 - m_2)n + \{-3, \dots, 3\}.$$

By Sturm comparison,  $v_2$  has at least one zero strictly between consecutive zeros of  $v_1$ ; consequently, the zeros of  $v_1, v_2$  are eventually alternating, for otherwise  $N$  would be unbounded, contrary to Lemma 1. Hence,

$$\mathcal{N}_t(\lambda_1, \lambda_2) \in N_2 - N_1 + (m_1 - m_2)n + \{-4, \dots, 3\}.$$

■

**Proposition 2** *In addition to the general hypotheses, let  $p \in L_\infty(0, \infty)$ ,  $\|p\|_\infty \leq \varepsilon$ , and*

$$\tilde{t} = t + p.$$

*Then the total spectral multiplicities of  $t$  and  $\tilde{t}$  satisfy the estimate*

$$\mathcal{N}_t(\lambda_1 + \varepsilon, \lambda_2 - \varepsilon) \leq \mathcal{N}_{\tilde{t}}(\lambda_1, \lambda_2) \leq \mathcal{N}_t(\lambda_1 - \varepsilon, \lambda_2 + \varepsilon).$$

*Proof.* For each self-adjoint operator  $A$  we have by the spectral theorem,

$$\mathcal{N}_A(\lambda - \delta, \lambda + \delta) = \dim\{u \in D(A); \|(A - \lambda)u\| < \delta\|u\|\},$$

for  $\lambda \in \mathbb{R}$  and  $\delta > 0$  (cf. [7, Chapter I, Theorem 9<sup>bis</sup>]). Hence the assertion follows since the perturbation  $p$  has operator norm  $\leq \varepsilon$  and  $\tilde{t}$  has the same domain as  $t$ . ■

## 2 Calculations

Given a periodic potential  $q$ , we wish to calculate an estimate for the number of eigenvalues of the singularly perturbed Sturm-Liouville operator

$$t_c = -\frac{d^2}{dr^2} + q(r) + \frac{c}{r^2} \quad \text{on } (0, \infty)$$

in an interval  $(\lambda_1, \lambda_2)$  compactly contained in a spectral gap of the unperturbed periodic operator. We consider coupling constants  $c \geq 3/4$ , so that the general hypotheses of the preceding section are satisfied ([18, Theorem 6.4 a]). Note that  $t_c$  remains non-oscillatory at 0 for  $c \in [-\frac{1}{4}, \frac{3}{4}]$ , and an analysis similar to the present is still possible. However, for the sake of simplicity of exposition, we omit this case, in which  $t_c$  is in the limit circle case at 0. Furthermore, we will assume that  $q$  is piecewise constant.



By means of oscillation theory, this question can be translated into studying a limit at infinity of the difference of the number of zeros of suitable solutions of the associated Sturm-Liouville differential equation with spectral parameter  $\lambda_1$  and  $\lambda_2$ . In a neighbourhood of the singular end-points 0 and  $\infty$ , the behaviour of this equation is dominated respectively by the perturbation and by the periodic part, so that sufficient qualitative information about the solutions can be inferred from abstract considerations. On a substantial intermediate interval, however, both the periodic background potential and the perturbation are of comparable size, and thus we resort to numerical calculation there. As it is one of our aims to handle fairly large values of the coupling constant, which implies a computation on large intervals, we wish the program to be both fast and robust. To this end, we use approximations, with an error controlled by Propositions 1 and 2, to replace the problem by one which can be solved explicitly in terms of a finite number of evaluations of standard trigonometric and exponential functions and arithmetical processes, without any further approximation.

The calculation proceeds as follows. The interval  $(\lambda_1, \lambda_2)$  is a subset of the  $n$ -th instability interval  $I_n$  of the unperturbed operator, for some  $n \in \mathbb{N}$ . Let  $\varepsilon > 0$  be such that  $[\lambda_1 - \varepsilon, \lambda_2 + \varepsilon] \subset I_n$  and  $\lambda_1 + \varepsilon < \lambda_2 - \varepsilon$ . Choose  $a > 0$  such that  $\inf_{r \in (0, a)} \{q(r) + \tilde{q}(r) - \lambda_2\} > 0$ , and  $m_j \in \mathbb{N}$  is fixed such that  $|\tilde{q}(r)| \leq \text{dist}(\lambda_j, \mathbb{R} \setminus I_n)$  for  $r \geq m_j \alpha$  and  $j \in \{1, 2\}$ . Here  $q$  is the given  $\alpha$ -periodic, piecewise constant background potential, and  $\tilde{q}$  is a piecewise constant perturbation differing from  $c/r^2$  by at most  $\varepsilon$  for  $r \in [a, \max_{i \in \{1, 2\}} m_j]$  (outside this interval we assume  $\tilde{q}(r) = c/r^2$  for simplicity). The effect on the spectrum of the transition from  $c/r^2$  to  $\tilde{q}$  is controlled by Proposition 2. On a technical level,  $\tilde{q}$  is constructed on each interval  $J$  where  $q$  is constant by equipartitioning the range  $\{c/r^2; r \in J\}$  into subintervals of length  $< 2\varepsilon$ , and assigning to  $\tilde{q}$  the centre value of each subinterval on its pre-image with respect to  $c/r^2$ . (The tempting idea of calculating Bessel functions of order  $\sqrt{c + \frac{1}{4}}$  to approximate the solutions proved inefficient and unreliable for large values of  $c$  and was therefore abandoned.)

The central part of the computation now consists of counting the number of zeros in  $(a, m_j \alpha]$  of the solution of the differential equation (2) with initial condition  $u(a) = 0$ . As the differential equation has piecewise constant coefficients, we have a finite sequence  $a = r_0 < r_1 < \dots < r_K = m_j \alpha$  such that it takes the form

$$-u'' + c_k u = 0$$

on  $(r_{k-1}, r_k)$ , with constants

$$c_k = (q + \tilde{q}) \left( \frac{r_k + r_{k-1}}{2} \right) - \lambda_j$$

for  $k \in \{1, \dots, K\}$ . Thus the values of the solution and its derivative at  $r_k$  are given by their values at  $r_{k-1}$  in terms of the transfer matrix

$$\begin{pmatrix} u \\ u' \end{pmatrix} (r_k) = \begin{pmatrix} C(c_k, r_k - r_{k-1}) & S(c_k, r_k - r_{k-1}) \\ c_k S(c_k, r_k - r_{k-1}) & C(c_k, r_k - r_{k-1}) \end{pmatrix} \begin{pmatrix} u \\ u' \end{pmatrix} (r_{k-1}),$$

where for  $r \in \mathbb{R}$

$$C(c, r) := \begin{cases} \cos(\sqrt{-c}r) \\ 1 \\ \cosh(\sqrt{c}r) \end{cases}, \quad S(c, r) := \begin{cases} \sin(\sqrt{-c}r)/\sqrt{c} \\ r \\ \sinh(\sqrt{c}r)/\sqrt{c} \end{cases} \quad \text{if } \begin{cases} c < 0 \\ c = 0 \\ c > 0 \end{cases}.$$

The values  $\begin{pmatrix} u \\ u' \end{pmatrix} (r_k)$ ,  $k \in \{1, \dots, K\}$ , are thus given by recursion from  $\begin{pmatrix} u \\ u' \end{pmatrix} (a) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ . As the calculation is performed within a spectral gap, i.e. an instability interval

of the periodic equation (1), the solution tends to grow exponentially. In order to avoid overflow errors, the solution vector  $\begin{pmatrix} u \\ u' \end{pmatrix}$  is renormalized when exceeding some fixed limit.

Moreover, the number of zeros of  $u$  in the half-open subinterval  $(r_{k-1}, r_k]$  can be deduced from the knowledge of these values alone. Indeed, the signs of  $u(r_k)$  and  $u(r_{k-1})$  (or  $u'(r_k)$ ,  $u'(r_{k-1})$  if  $r_k$  or  $r_{k-1}$  is a zero of  $u$ ) show whether the total number of zeros in  $(r_{k-1}, r_k]$  is even or odd. Once this is known, this number can easily be determined: for  $c \geq 0$  the equation is disconjugate, and hence  $u$  has at most one zero, and in case  $c < 0$  the solution is a trigonometric function, with either

$$2\lfloor \frac{(r_k - r_{k-1})\sqrt{c}}{2\pi} + \frac{1}{2} \rfloor, \text{ or } 2\lfloor \frac{(r_k - r_{k-1})\sqrt{c}}{2\pi} \rfloor + 1$$

zeros. Here  $\lfloor \dots \rfloor$  stands for the floor function (or integer part).

Finally, the zero counts thus obtained for each  $\lambda_j$  are used to calculate estimates for the number of eigenvalues in  $(\lambda_1, \lambda_2)$  according to Proposition 1.

The program is set up to count the number of zeros of solutions not only for two values, but for an array of values  $(\lambda_1, \dots, \lambda_N)$  of the spectral parameter simultaneously, thus giving estimates for the number of eigenvalues in all intervals delimited by pairs of such values. The left-hand end-point  $a$  is chosen to be the same for all  $\lambda_j$ , but the  $m_j$  are selected individually. The approximation of the perturbation by a piecewise constant function is independent of the particular value of  $\lambda_j$  and thus can be done once for all of these values.

The resulting histogram representation of eigenvalue counts is particularly suitable in view of the properties of the approximations involved. Indeed, the approximation of Proposition 2 moves each eigenvalue by at most  $\varepsilon$ , and thus an eigenvalue removed from any one subinterval of the histogram will appear in a neighbouring subinterval. Furthermore, the error estimate of Proposition 1 holds uniformly for all intervals  $(\lambda_k, \lambda_l)$ ,  $k, l \in \{1, \dots, N\}$ ,  $k < l$ . Thus the error committed by our approximations does not accumulate across the histogram, but rather tends to compensate between adjacent subintervals, providing a quite reliable picture of the distribution of eigenvalues.

For convenience, given the number  $n$ , the program determines  $I_n$  to the desired precision automatically (cf. Remark after Lemma 1), and provides an equipartition  $\lambda_1 < \dots < \lambda_N$  for a specified  $N$ .

The method detailed above was implemented as a purpose-written Fortran 77 prototype code, running under the UNIX operating system. Only built-in Fortran standard functions and double-precision arithmetic were used. Round-off errors cannot be ruled out a priori, but their influence is reduced by the fact that our scheme relies on counting integer quantities; furthermore, it is apparent from the excellent correspondence of the numerical results with the asymptotics even in the most prolonged computations (such as the case of  $c = 10^{14}$  in Table 2) that round-off errors were negligible throughout compared to the theoretical error of Proposition 1.

**Example 1** Consider the Meissner potential (cf. [10])

$$q(r) = \begin{cases} 1, & r \in (-\frac{\pi}{2}, \frac{\pi}{2}] + 2\pi\mathbb{Z}, \\ -1, & r \in (\frac{\pi}{2}, \frac{3\pi}{2}] + 2\pi\mathbb{Z}. \end{cases}$$

The first spectral gap of the corresponding Sturm-Liouville operator  $t_0$  contains the interval  $[-0.527035551, 0.560730804]$  (and these numbers are within  $10^{-8}$  of its end-points). The resulting estimated eigenvalue counts for the perturbed operator for

various values of the coupling constant  $c$  are given in Table 1. The approximation governed by Proposition 2 was computed with  $\varepsilon = 0.001$ . Corresponding values for the second gap  $[0.765821159, 1.466455365]$  are collected in Table 2.

Table 1: Calculated estimates for the number of eigenvalues in subintervals of the first spectral gap for Example 1, for a variety of coupling constants  $c$ .  $\lambda$  values indicate the upper and lower bounds of each subinterval.

$\lambda$	$c = 1$	100	$10^4$	$10^6$	$10^8$	$10^{10}$	$10^{12}$	$10^{14}$
-0.5260	1	16	159	1583	15830	158300	1583006	15830047
-0.4175	1	2	13	127	1276	12758	127582	1275825
-0.3089	0	0	5	60	590	5902	59020	590193
-0.2003	0	1	4	35	358	3580	35798	357985
-0.0917	0	0	3	25	247	2467	24664	246633
0.0168	0	0	1	18	183	1831	18315	183156
0.1254	0	0	2	15	143	1430	14295	142943
0.2340	0	0	1	11	116	1155	11557	115575
0.3426	0	0	1	10	96	960	9596	95957
0.4512	0	0	1	8	81	813	8132	81326
0.5597								

It is striking that over the whole range of values of  $c$  considered, the number of eigenvalues in each subinterval scales very precisely in proportion to  $\sqrt{c}$ . This observation has led us to conjecture that for all values  $c$  of the coupling constant, the number of eigenvalues in any given subinterval of a spectral gap is proportional to  $\sqrt{c}$ , with a very small error. We shall collect further evidence for this conjecture in the following section, calculating estimated eigenvalue counts for *small* values of the coupling constants and  $\lambda$ -intervals close to the lower end of the first spectral gap.

### 3 Comparison with the asymptotic formula for the large-coupling limit

In the large coupling constant limit, asymptotic scaling  $\sim \sqrt{c}$  for the eigenvalue density is well known. Indeed, for an  $\alpha$ -periodic  $q$ , and perturbation  $\tilde{q}(r) \sim \frac{c}{r^2}$  ( $r \rightarrow \infty$ ), the number of eigenvalues of  $t_c$  in an interval  $(\lambda_1, \lambda_2)$  compactly contained in a spectral gap of  $t_0$  satisfies

$$N(\lambda_1, \lambda_2; c) \sim \frac{\sqrt{c}}{\pi\alpha} \int \int \chi_{\lambda_1, \lambda_2}(\lambda, r) dk(\lambda) dr \quad (3)$$

asymptotically as  $c \rightarrow \infty$  [15]; here  $\chi_{\lambda_1, \lambda_2}$  is the characteristic function of the set  $\{(\lambda, r) \in \mathbb{R} \times (0, \infty); \lambda_1 \leq \lambda + \frac{1}{r^2} \leq \lambda_2\}$ , and  $k$  denotes the so-called quasimomentum

Table 2: Calculated estimates for the number of eigenvalues in subintervals of the second spectral gap for Example 1 (cf. Table 1).

$\lambda$	$c = 1$	100	$10^4$	$10^6$	$10^8$	$10^{10}$	$10^{12}$	$10^{14}$
0.7668	0	4	47	469	4682	46809	468093	4680926
0.8666	0	1	8	76	773	7737	77368	773682
0.9664	1	0	4	45	444	4438	44379	443782
1.0662	0	1	4	31	305	3059	30588	305884
1.1660	0	0	2	22	232	2306	23055	230549
1.2658	0	0	1	19	183	1834	18341	183408
1.3657	0	0	2	16	151	1513	15130	151298
1.4655								

of the unperturbed periodic equation. The quasimomentum is a continuous non-decreasing function related to the discriminant by  $D(\lambda) = 2 \cos(k(\lambda))$  for  $\lambda$  inside the spectral bands, and it is constant in the spectral gaps of  $t_0$ .

For fixed  $\lambda \in \mathbb{R}$ , the condition  $\lambda_1 \leq \lambda + \frac{1}{r^2} \leq \lambda_2$  translates into

$$\sqrt{\frac{1}{\lambda_2 - \lambda}} \leq r \leq \sqrt{\frac{1}{\lambda_1 - \lambda}}.$$

Assume that  $\lambda_1, \lambda_2$  lie in the first spectral gap  $(M_1, \mu_1)$ ; then, as  $dk(\lambda) = 0$  outside the bands, such that the integration with respect to  $\lambda$  extends over values below  $\lambda_1$  only, we find by carrying out the integration over  $r$  and substituting  $\kappa$  for  $k(\lambda)$ ,

$$N(\lambda_1, \lambda_2; c) \sim \frac{\sqrt{c}}{\pi\alpha} \int_0^\pi \left( \frac{1}{\sqrt{\lambda_1 - k^{-1}(\kappa)}} - \frac{1}{\sqrt{\lambda_2 - k^{-1}(\kappa)}} \right) d\kappa \quad (c \rightarrow \infty);$$

here  $k^{-1}(\kappa) = D^{-1}(2 \cos(\kappa))$ , for  $\kappa \in [0, \pi]$ , represents the inverse of  $k$  in the first spectral band. (An attempt to carry out the integration over  $\lambda$  first in (3) resulted in a numerically subtle expression, which shows that Fubini's theorem may be hazardous in practice.) Figure 1 gives an impression of the density of eigenvalues in a gap.

It can be shown [14] that for  $c$  larger than the critical constant  $c_{\text{crit}} = \frac{\alpha^2}{4|D'|(\mu_n)}$  there are infinitely many eigenvalues of  $t_c$  in  $(M_n, \mu_n)$ , accumulating at  $M_n$  like

$$\frac{\sqrt{\frac{c}{c_{\text{crit}}} - 1}}{4\pi} |\ln(\lambda - M_n)|.$$

On the other hand,

$$\frac{\sqrt{c}}{\pi\alpha} \int_0^\pi \frac{d\kappa}{\sqrt{\lambda - k^{-1}(\kappa)}} \sim \frac{\sqrt{c/c_{\text{crit}}}}{4\pi} |\ln(\lambda - M_1)|$$

as  $\lambda \rightarrow M_1$ , and this is the leading term of the above asymptotic formula in the limit  $c \rightarrow \infty$ .

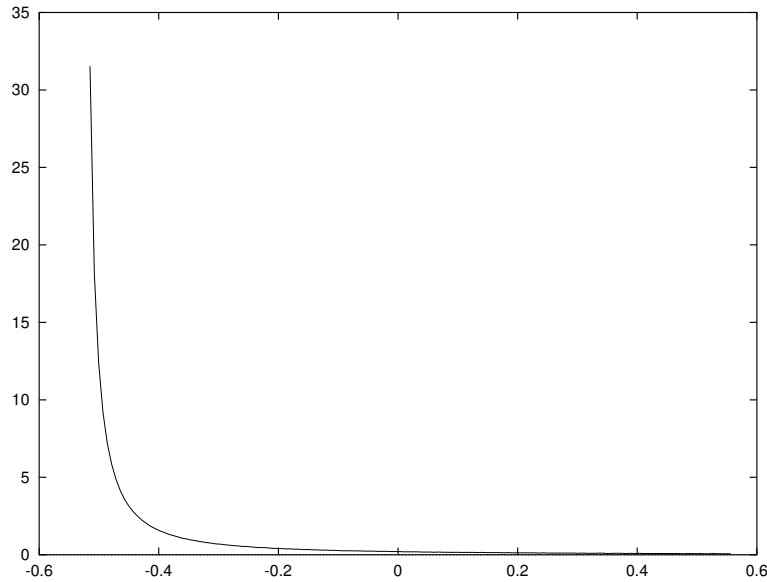


Figure 1: Density of eigenvalues in the first gap for Example 1

The conjecture formulated at the end of the preceding section can now be recast in the following way: *The expression on the right-hand side of the asymptotic formula (3) gives a very accurate description of the number of eigenvalues in  $(\lambda_1, \lambda_2)$  even for small values of  $c$ .*

In order to test this conjecture, we compared values of this expression with calculated eigenvalue estimates for subintervals of the first spectral gap and a wide range of coupling constants  $c$ , including values below  $c_{\text{crit}}$ . In Example 1, the critical constant for the lower end of the first spectral gap is  $c_{\text{crit}} \approx 0.051$ , which gave rise to a large number of eigenvalues. For our present purpose, we now shorten the period to 1, in order to obtain a sufficiently large critical constant, namely  $c_{\text{crit}} \approx 7.5$ . For further evidence, we also studied a non-symmetric, but otherwise arbitrarily chosen potential.

### Example 2

$$q_1(r) := \begin{cases} 1, & r \in (-0.25, 0.25] + \mathbb{Z}, \\ -1, & r \in (0.25, 0.75] + \mathbb{Z}; \end{cases}$$

$$q_2(r) := \begin{cases} 1, & r \in (0, 0.4] + \mathbb{Z}, \\ 2.9, & r \in (0.4, 0.63] + \mathbb{Z}, \\ -4.7, & r \in (0.63, 0.91] + \mathbb{Z}, \\ 0.3, & r \in (0.91, 1] + \mathbb{Z}. \end{cases}$$

In both instances, we counted the eigenvalues in a series of subintervals of the first spectral gap, with fixed right-hand end-point, and the left-hand end-point approaching the lower edge of the gap (where eigenvalues accumulate). In order to control the error introduced by the approximation of the perturbation by a piecewise constant function, we also calculated the values for the two intervals of length  $\varepsilon$ , immediately adjacent to the left and right of the interval under consideration, and chose the accuracy  $\varepsilon$  so small that at most one eigenvalue (as it turned out, no eigenvalues in most cases) was observed there.

The integral appearing in the asymptotic formula was calculated using NAG integrator D01AJF, and interpolating from a table of the discriminant to obtain the inverse  $k^{-1}$  in the integrand.

Table 3: Maximal differences between calculated estimates for the number of eigenvalues in subintervals of the first spectral gap and corresponding values of the asymptotic formula, together with maximal numbers of eigenvalues in these intervals. Left-hand columns are for  $q_1$ , right-hand columns for  $q_2$ ; coupling constants  $c = 1, 4, 4^2, \dots, 4^{10}$  (upper sections), resp.  $c = 1, 5, \dots, 5^6$  (lower sections) are taken into account.

$\lambda$ -interval	max. diff.	max. # of evs.	$\lambda$ -interval	max. diff.	max. # of evs.
(9.9, 10)	-0.7972	8	(9.8, 10)	0.8862	9
(9.7, 10)	0.7818	26	(9.4, 10)	-0.9248	29
(9.5, 10)	0.8712	52	(8.8, 10)	-0.9153	72
(9.3, 10)	0.7557	103	(8.4, 10)	-0.8358	122
(9.28, 10)	0.8301	114	(8.2, 10)	-0.7600	172
(9.26, 10)	0.8384	130	(8.1, 10)	-0.6867	231
(9.24, 10)	1.0353	160	(8.08, 10)	0.6928	258
(9.23, 10)	0.8062	210	(8.06, 10)	-1.0178	320
(9.228, 10)	0.8962	263	(8.059, 10)	-0.8701	326
(9.2278, 10)	0.8052	282	(8.055, 10)	-1.0740	369
(9.2276, 10)	-0.7899	357	(8.053, 10)	-1.0794	427
(9.22759, 10)	-0.7810	383	(8.0529, 10)	-1.1421	434
(9.2276, 10)	-0.7437	44	(8.055, 10)	-0.8027	45
(9.22759, 10)	-0.8707	47	(8.053, 10)	0.5820	52
(9.227588, 10)	-0.9184	48	(8.0527, 10)	0.5608	55
(9.227586, 10)	-0.9897	50	(8.0523, 10)	-0.6907	63
(9.227585, 10)	-1.0451	51	(8.05219, 10)	-0.7153	78
(9.227584, 10)	1.1358	53	(8.052183, 10)	-0.5901	88

For each subinterval of the first spectral gap ((9.22758284624, 10.50068969088) for  $q_1$  and (8.05218126855, 11.13944279771) for  $q_2$ ) we determined the maximal absolute difference between the estimated number of eigenvalues and the value of the asymptotic formula, for coupling constants  $c \in \{4^l; l \in \{0, \dots, 10\}\}$ , or (in the case of subintervals very close to the lower edge of the gap, for which  $\varepsilon$  must be taken very small, resulting in a correspondingly high computation cost)  $c \in \{5^l; l \in \{0, \dots, 6\}\}$ . The results are collected in Table 3. We point out that the observed absolute differences are less than 2 in all cases, well below the error of the estimate of Proposition 1, and hence strikingly corroborate the above conjecture.

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# A study of the Laguerre-Hahn affine functionals on the unit circle.

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## Abstract

The aim of this paper is to study the Laguerre-Hahn affine functionals on the unit circle, and to establish the relation with the semiclassical ones. We prove that both classes are different and we characterize those functionals which are Laguerre-Hahn affine and which are not semiclassical. Some examples are given.

Finally, we present several modifications of the functional that preserve the Laguerre-Hahn affine character.

*Key words and phrases:* Orthogonal polynomials, Semiclassical functionals, Laguerre-Hahn affine functionals.

*Mathematics Subject Classification:* 33C47; 42C05.

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## 1 Introduction

The study of the differential properties of the Stieltjes function associated with quasi-definite linear functionals of Hankel type (see [7]), has allowed to obtain a classification of the orthogonal polynomial families. Thus, the semiclassical orthogonal polynomials appear in a natural way in relation with Stieltjes functions which are the solutions of linear ordinary differential equations with polynomial coefficients, (see [12]). A larger class corresponds to the Laguerre-Hahn functionals characterized by the fact that the Stieltjes function satisfies a Riccati differential equation with polynomial coefficients. A complete study of this type of functionals appears in [10] and [2]. Moreover, the semiclassical functionals are identified with the subclass of the Laguerre-Hahn functionals, called the

Laguerre-Hahn affine class, characterized by the fact that in the Riccati differential equation the quadratic term does not appear.

For linear functionals which are quasi-definite and hermitian we can consider the formal series  $G(z) = \sum_{n=-\infty}^{\infty} \overline{c_n} z^n$ , where  $c_n$  is the  $n$ -th moment for the functional. The study of the differential properties of  $G$  allows us to obtain a similar classification for the orthogonal families on the unit circle.

Orthogonal polynomials with respect to linear functionals such that  $G$  satisfies an homogeneous linear differential equation of first order with polynomial coefficients are called semiclassical polynomials. Several authors have studied such semiclassical orthogonal polynomials on the unit circle and different characterizations and properties are well-known (see [1], [14], [11] and [4]).

The Laguerre-Hahn functionals and the Laguerre-Hahn affine functionals are introduced in [14]. Nevertheless the study of this kind of functionals and the properties of the corresponding orthogonal polynomials have not been developed yet. Moreover, the relation between the semiclassical and the Laguerre-Hahn affine functionals has not been established.

In the present paper, our aim is to study the Laguerre-Hahn affine functionals and to establish its relation with the semiclassical functionals. The organization of the paper is the following:

In section 2 we introduce the Laguerre-Hahn affine functionals on the unit circle and we prove that there exist Laguerre-Hahn affine functionals which are not semiclassical.

In section 3 we study the differential properties of the formal series  $F(z) = c_0 + 2 \sum_{n=1}^{\infty} \overline{c_n} z^n$ , and we solve a problem posed in [1]: If the formal series  $F$  satisfies a certain type of non-homogeneous linear differential equation of first order with polynomial coefficients; is the corresponding functional a semiclassical one?

In section 4, we present some modifications of a functional that preserve the Laguerre-Hahn affine character. Finally, in section 5, given a Laguerre-Hahn affine functional such that the measure belongs to the Szegő class, we deduce that the reversed of the monic orthogonal polynomial of degree  $n$  satisfies a Riccati differential equation with coefficients depending on  $n$ .

## 2 Laguerre-Hahn affine functionals

Let  $\Lambda$  be the space of Laurent polynomials, that is,  $\Lambda = \text{span}\{z^k : k \in \mathbb{Z}\}$  and let  $\mathcal{L} : \Lambda \rightarrow \mathbb{C}$  be a linear moment functional which is quasi-definite and hermitian. If we denote the moments of the functional  $\mathcal{L}$  by  $c_n = \mathcal{L}(z^n)$  for every integer  $n \in \mathbb{Z}$ , we say that  $\mathcal{L}$  is hermitian if  $\forall n \geq 0$   $c_{-n} = \overline{c_n}$ , and  $\mathcal{L}$  is said to be quasi-definite (positive definite) if the principal submatrices of the moment matrix are nonsingular (positive definite), i.e,

$$\forall n \geq 0, \quad \Delta_n = \det(c_{i-j})_{i,j=0}^n \neq 0 (> 0).$$

When  $\mathcal{L}$  is positive definite it is well-known (see [8]) that there exists a finite positive Borel measure  $\mu$  on  $[0, 2\pi]$  such that

$$c_n = \int_0^{2\pi} z^n d\mu(\theta), \quad z = e^{i\theta}.$$

Next we recall the following definitions (see [14]).

**Definition 1** *A quasi-definite and hermitian linear functional is said to be Laguerre-Hahn affine if there exist polynomials  $A$ ,  $B$ , and  $H$  with  $A \neq 0$  such that the formal series*

$$G(z) = \sum_{n=-\infty}^{\infty} \overline{c_n} z^n$$

*and its formal derivative  $G'(z) = \sum_{n=-\infty}^{\infty} n \overline{c_n} z^{n-1}$  satisfies*

$$A(z)G'(z) + B(z)G(z) + H(z) = 0. \quad (1)$$

*If  $G$  satisfies a differential equation of the form (1) with  $H = 0$ , the functional is called semiclassical.*

*The formal series  $G$  is said to be rational if there exist polynomials  $C$  and  $D$  such that*

$$C(z)G(z) = D(z) \quad \text{with } C \neq 0.$$

For example, if we consider the quasi-definite and hermitian linear functional  $\mathcal{L}$  with moments  $c_n = (-1)^n(1 + \frac{n}{2})$  for  $n \geq 0$ , then the formal series  $G$  satisfies the equation

$$(z+1)^2 G'(z) + (z+1)(z+3)G(z) + (z+1) = 0,$$

and also

$$(z+1)^2 G(z) = -z.$$

Since in the positive definite case the relation between the Radon-Nikodym derivative of the measure  $\mu$  and the series  $G$  is  $\mu'(\theta) = G(e^{i\theta})$  a.e. in  $[0, 2\pi)$ , it is natural to introduce the semiclassical and the Laguerre-Hahn affine functionals by (1).

It is clear that the differential equation satisfied by  $G$  is not unique. However, it is not clear if there exist Laguerre-Hahn affine functionals which are not semiclassical. So, in order to give an answer to this question, next we present some properties of the Laguerre-Hahn affine functionals.

**Theorem 1** *Let  $\mathcal{L}$  be a quasi-definite and hermitian linear functional such that the corresponding  $G$  satisfies (1). Then either*

$$A(z)\overline{H}\left(\frac{1}{z}\right) + z^2\overline{A}\left(\frac{1}{z}\right)H(z) \neq 0, \text{ and } A(z)\overline{B}\left(\frac{1}{z}\right) + z^2\overline{A}\left(\frac{1}{z}\right)B(z) \neq 0, \quad (2)$$

or

$$\begin{aligned} A(z)\overline{H}\left(\frac{1}{z}\right) + z^2\overline{A}\left(\frac{1}{z}\right)H(z) &= A(z)\overline{B}\left(\frac{1}{z}\right) + z^2\overline{A}\left(\frac{1}{z}\right)B(z) = \\ B(z)\overline{H}\left(\frac{1}{z}\right) - \overline{B}\left(\frac{1}{z}\right)H(z) &= 0. \end{aligned} \quad (3)$$

If (2) holds then  $G$  is rational and  $\mathcal{L}$  is also semiclassical.

Proof: From (1), taking into account  $\overline{G}\left(\frac{1}{z}\right) = G(z)$  and  $\overline{G'}\left(\frac{1}{z}\right) = -z^2G'(z)$ ,

$$-z^2\overline{A}\left(\frac{1}{z}\right)G'(z) + \overline{B}\left(\frac{1}{z}\right)G(z) + \overline{H}\left(\frac{1}{z}\right) = 0. \quad (4)$$

Combining both relations (1) and (4) we get

$$\left(A(z)\overline{H}\left(\frac{1}{z}\right) + z^2\overline{A}\left(\frac{1}{z}\right)H(z)\right)G'(z) + \left(B(z)\overline{H}\left(\frac{1}{z}\right) - \overline{B}\left(\frac{1}{z}\right)H(z)\right)G(z) = 0, \quad (5)$$

$$\left(A(z)\overline{B}\left(\frac{1}{z}\right) + z^2\overline{A}\left(\frac{1}{z}\right)B(z)\right)G'(z) + \left(B(z)\overline{H}\left(\frac{1}{z}\right) - \overline{B}\left(\frac{1}{z}\right)H(z)\right)G(z) = 0, \quad (6)$$

$$\left(A(z)\overline{B}\left(\frac{1}{z}\right) + z^2\overline{A}\left(\frac{1}{z}\right)B(z)\right)G(z) + \left(A(z)\overline{H}\left(\frac{1}{z}\right) + z^2\overline{A}\left(\frac{1}{z}\right)H(z)\right)G'(z) = 0. \quad (7)$$

Therefore, by (5)-(7), (2) and (3) hold.

If (2) holds, it is straightforward from (5) and (7) that  $\mathcal{L}$  is semiclassical and  $G$  is rational.  $\square$

Note if  $H = 0$ , i.e.  $\mathcal{L}$  is semiclassical, then (3) holds.

Since the Theorem 1 gives necessary conditions we are interested in the characterization of functionals  $\mathcal{L}$  satisfying (1) and the second case (3) in Theorem 1.

Next we present two examples of functionals in this situation, for which it is well-known that they are semiclassical, (see [14]).

### Examples 1.

- (1) Let  $\mathcal{L}$  be the positive definite functional associated with the measure  $d\mu(\theta) = (1 - c)\frac{d\theta}{2\pi} + cd\delta_{-1}(\theta)$ , where  $\frac{d\theta}{2\pi}$  is the normalized Lebesgue measure on  $[0, 2\pi]$ ,  $c$  is a real number,  $c \in (0, 1)$ , and  $d\delta_{-1}(\theta)$  is the Dirac

delta measure supported at  $-1$ . Hence  $G(z) = \sum_{n=-\infty}^{\infty} \overline{c_n} z^n$  with  $c_0 = 1$  and  $c_n = (-1)^n c$  for  $n \neq 0$ , and it is easy to deduce that  $G$  satisfies (1). In fact

$$(1 - z^2)G'(z) + 2G(z) + 2(c - 1) = 0.$$

- (2) Let  $\mathcal{L}$  be the positive definite functional associated with the measure  $d\mu(\theta) = (1 - c)\frac{d\theta}{2\pi} + cd\delta_1(\theta)$ , where  $\frac{d\theta}{2\pi}$  is the normalized Lebesgue measure on  $[0, 2\pi]$ ,  $c$  is a real number,  $c \in (0, 1)$ , and  $d\delta_1(\theta)$  is the Dirac delta measure with mass point at 1. Hence  $G(z) = \sum_{n=-\infty}^{\infty} \overline{c_n} z^n$  with  $c_0 = 1$  and  $c_n = c$  for  $n \neq 0$ . Then  $G$  satisfies

$$(1 - z^2)G'(z) - 2G(z) + 2(1 - c) = 0.$$

**Theorem 2** *Let  $\mathcal{L}$  be a quasi-definite and hermitian linear functional such that*

$$A(z)G'(z) + B(z)G(z) + H(z) = 0, \text{ with } A \neq 0, \text{ and } H \neq 0.$$

*If (3) holds, that is,*

$$\begin{aligned} A(z)\overline{H}\left(\frac{1}{z}\right) + z^2\overline{A}\left(\frac{1}{z}\right)H(z) &= A(z)\overline{B}\left(\frac{1}{z}\right) + z^2\overline{A}\left(\frac{1}{z}\right)B(z) = \\ B(z)\overline{H}\left(\frac{1}{z}\right) - \overline{B}\left(\frac{1}{z}\right)H(z) &= 0, \end{aligned}$$

*then*

*$\mathcal{L}$  is semiclassical if and only if  $G$  is rational.*

*Moreover, if  $p = \deg(A)$ ,  $q = \deg(B)$ ,  $s = \deg(H)$ , and  $\alpha, \beta, \gamma$  are the multiplicities of 0 as a zero of  $A, B$ , and  $H$  respectively, then*

$$p \geq 1 \text{ and } p - 2 + \alpha = q + \beta = s + \gamma.$$

Proof: If  $\mathcal{L}$  is semiclassical, there exist polynomials  $U$  and  $V$  such that

$$UG' + VG = 0, \text{ with } U, V \neq 0. \quad (8)$$

From relation (1) we get  $UAG' + UBG + UH = 0$ , and taking into account (8), we obtain  $(-VA + UB)G + UH = 0$ , i.e.  $G$  is rational.

Conversely, assume that there exist polynomials  $C$  and  $D$  different from zero such that  $CG = D$ . Since  $DAG' + DBG + DH = 0$ , we get that

$DAG' + (DB + HC)G = 0$ , with  $DA \neq 0$ . Hence  $\mathcal{L}$  is semiclassical.

From (3) we obtain, if  $B \neq 0$ ,

$$-\frac{A(z)}{z^2 \overline{A}(\frac{1}{z})} = \frac{B(z)}{\overline{B}(\frac{1}{z})} = \frac{H(z)}{\overline{H}(\frac{1}{z})}. \quad (9)$$

We can write  $A(z) = z^\alpha \widehat{A}(z)$ , with  $\widehat{A}(0) \neq 0$ ,  $B(z) = z^\beta \widehat{B}(z)$ , with  $\widehat{B}(0) \neq 0$ , and  $H(z) = z^\gamma \widehat{H}(z)$ , with  $\widehat{H}(0) \neq 0$ . Therefore (9) can be rewritten as follows

$$-\frac{z^{p-2+\alpha} \widehat{A}}{\widehat{A}^*} = \frac{z^{q+\beta} \widehat{B}}{\widehat{B}^*} = \frac{z^{s+\gamma} \widehat{H}}{\widehat{H}^*}. \quad (10)$$

(Recall that the reversed polynomials are defined by  $P^*(z) = z^n \overline{P}(\frac{1}{z})$  if  $\deg(P) = n$ , see [8].)

If  $p = 0$ , then  $\alpha = 0$  and  $A(z) = a_0$ . Thus (10) becomes

$$-\frac{a_0}{\overline{a_0}} = \frac{z^{q+\beta+2} \widehat{B}}{\widehat{B}^*} = \frac{z^{s+\gamma+2} \widehat{H}}{\widehat{H}^*}.$$

Since  $\frac{a_0}{\overline{a_0}} \neq 0$  and  $q + \beta + 2 \geq 2$ , and  $s + \gamma + 2 \geq 2$ , we get a contradiction.

Next we prove that  $p = 1$  implies  $\alpha = 1$ .

Indeed, if  $\alpha = 0$  then  $A(z) = a_1 z + a_0$  with  $a_0 \neq 0$  and  $a_1 \neq 0$ . Hence (10) is

$$-\frac{z^{-1}(a_1 z + a_0)}{\overline{a_1} + \overline{a_0} z} = \frac{z^{q+\beta} \widehat{B}}{\widehat{B}^*} = \frac{z^{s+\gamma} \widehat{H}}{\widehat{H}^*},$$

and therefore

$$-\frac{(a_1 z + a_0)}{\overline{a_1} + \overline{a_0} z} = \frac{z^{q+\beta+1} \widehat{B}}{\widehat{B}^*} = \frac{z^{s+\gamma+1} \widehat{H}}{\widehat{H}^*}$$

holds. Since  $\frac{a_0}{\overline{a_1}} \neq 0$  and  $q + \beta + 1 \geq 1$  and  $s + \gamma + 1 \geq 1$ , the preceding relation is not true. Thus,  $\alpha = 1$  and  $q + \beta = s + \gamma = 0$ .

Finally, if  $p \geq 2$ , from (10) we deduce that  $p - 2 + \alpha = q + \beta = s + \gamma$ .  $\square$

Now we are able to show that on the unit circle the class of semiclassical functionals and the class of Laguerre-Hahn affine functionals are different, in contrast to the well-known result on the real line (see [10] and [12]).

**Corollary 1** *The only Laguerre-Hahn affine functionals which are not semiclassical are those such that the series  $G$  is not rational and satisfies*

$$A(z)G'(z) + B(z)G(z) + H(z) = 0, \text{ with } A \neq 0, H \neq 0, \quad (11)$$

and (3):

$$\begin{aligned} A(z)\overline{H}(\frac{1}{z}) + z^2 \overline{A}(\frac{1}{z})H(z) &= A(z)\overline{B}(\frac{1}{z}) + z^2 \overline{A}(\frac{1}{z})B(z) = \\ &= B(z)\overline{H}(\frac{1}{z}) - \overline{B}(\frac{1}{z})H(z) = 0. \end{aligned}$$



Therefore in the positive definite case, the only Laguerre-Hahn affine measures which are not semiclassical are those ones for which the series  $G$ , that is the Radon-Nikodym derivative of the measure a.e. on the unit circle, is not rational and satisfies (11) and (3).

Next, using Corollary 1, we present some examples of Laguerre-Hahn affine functionals which are not semiclassical.

### Examples 2.

- (1) Let  $L$  be a semiclassical functional with moments  $\{c_n\}$ , and such that the series  $G$  is not rational and satisfies  $AG' + BG = 0$ . Let us consider the functional  $\mathcal{L}$  with moments  $d_n = c_n \ \forall n \neq 0$  and  $d_0 = c_0 + 1$ , that is,  $\mathcal{L}$  is the sum of functional  $L$  and the functional  $L_1$  associated with the normalized Lebesgue measure. If we denote  $\mathcal{G} = \sum_{n=-\infty}^{\infty} \overline{d_n} z^n$ , taking into account that  $\mathcal{G} = G + 1$ , then

$$A\mathcal{G}' + B\mathcal{G} - B = 0.$$

From Corollary 1 we deduce that  $\mathcal{L}$  is a Laguerre-Hahn affine functional which is not semiclassical.

As a particular case of the above situation, if  $L$  is the Jacobi functional corresponding to the measure  $d\mu(\theta) = (\cos(\frac{\theta}{2}))|\sin(\frac{\theta}{2})|\frac{d\theta}{2\pi}$ , then the formal series  $G$  is not rational and  $G$  satisfies  $z(z^2 - 1)G'(z) - (z^2 + 1)G(z) = 0$ , (see [11]). Then if  $\mathcal{L} = L + L_1$  the corresponding series  $\mathcal{G}$  satisfies  $z(z^2 - 1)\mathcal{G}'(z) - (z^2 + 1)\mathcal{G}(z) + (z^2 + 1) = 0$ .

- (2) Let  $L$  be a semiclassical functional,  $\{c_n\}$  the sequence of its moments, and such that the series  $G$  is not rational and satisfies  $AG' + BG = 0$ . We consider a quasi-definite functional  $L_k$  such that the series  $G_k$  is  $G_k(z) = \sum_{n=-k}^k \overline{b_n} z^n$ , and we assume that  $AG'_k + BG_k \neq 0$ . If  $\mathcal{L} = L + L_k$  and we denote its formal series by  $\mathcal{G}$ , then  $\mathcal{G} = G + G_k$ . When  $\mathcal{L}$  is quasi-definite  $\mathcal{L}$  is a Laguerre-Hahn affine functional satisfying  $A\mathcal{G}' + B\mathcal{G} - (AG'_k + BG_k) = 0$ , i.e.,

$$z^{k+1}A\mathcal{G}' + z^{k+1}B\mathcal{G} + H = 0,$$

with  $H = -z^{k+1}(AG'_k + BG_k)$ . From Corollary 1 we deduce that  $\mathcal{L}$  is not semiclassical.

## 3 Properties of the associated formal series

$$F(z) = c_0 + 2 \sum_{n=1}^{\infty} \overline{c_n} z^n.$$

In this section we study the differential properties of the formal series  $F(z)$  associated with a Laguerre-Hahn affine functional, i.e. associated with  $G(z) = \frac{F(z) + \overline{F}(\frac{1}{z})}{2}$ . Notice that in the positive definite case  $F$  is the Carathéodory function associated with the measure. Indeed we study the polynomial coefficients of the differential equation in order to obtain Laguerre-Hahn affine functionals.

**Theorem 3** *If  $\mathcal{L}$  is a Laguerre-Hahn affine functional such that  $G$  satisfies (1), then there exists a polynomial  $D$  such that the associated formal series  $F$  satisfies*

$$A(z)F'(z) + B(z)F(z) + 2H(z) + D(z) = 0, \quad (12)$$

and

$$A(z)(\overline{F}(\frac{1}{z}))' + B(z)\overline{F}(\frac{1}{z}) - D(z) = 0. \quad (13)$$

Let us denote by  $p, q, r$  and  $s$  the degrees of  $A, B, 2H + D$  and  $D$  respectively and by  $\alpha, \beta, \rho$  and  $\sigma$  the multiplicities of 0 as a zero of  $A, B, 2H + D$ , and  $D$ , respectively. Then one of the following conditions holds:

(i)  $F$  is rational.

(ii)

$$-\frac{A(z)}{z^2 \overline{A}(\frac{1}{z})} = \frac{B(z)}{\overline{B}(\frac{1}{z})} = \frac{2H(z) + D(z)}{-\overline{D}(\frac{1}{z})} = \frac{-D(z)}{2\overline{H}(\frac{1}{z}) + \overline{D}(\frac{1}{z})}, \quad (14)$$

and  $p + \alpha - 2 = q + \beta = s + \rho = r + \sigma = k$ .

Moreover, in case (ii) we have that  $2H + D$  and  $D$  have the same unimodular zeros, and there exist a natural number  $n_0$ , with  $0 \leq n_0 \leq s - \sigma$ , and a complex number  $\chi$  with  $|\chi| = 1$  such that

$$\omega(z) = -\frac{z^{p-2}A(z)}{A^*(z)} = \frac{z^q B(z)}{B^*(z)} = \frac{z^s(2H(z) + D(z))}{-D^*(z)} = \frac{-z^r D(z)}{(2H + D)^*(z)}$$

is given by

$$\omega(z) = \begin{cases} \chi z^k & \text{if } n_0 = 0, \\ \chi \prod_{j=1}^{n_0} \frac{(z - \beta_j)}{(1 - \overline{\beta_j} z)} z^k & \text{if } n_0 > 0. \end{cases}$$

In the last case,  $\beta_j$  are common zeros of  $A, B, 2H + D$  and  $D$  such that  $\beta_j \neq 0$  and  $|\beta_j| \neq 1$  for  $j = 1, \dots, n_0$ .

Proof: Since  $G(z) = \frac{F(z) + \overline{F}(\frac{1}{z})}{2}$ , we can write (1) as follows:

$$A(z) \left( F'(z) + (\overline{F}(\frac{1}{z}))' \right) + B(z) \left( F(z) + \overline{F}(\frac{1}{z}) \right) + 2H(z) = 0.$$

Let us consider the polynomial part of  $A(z)(\overline{F}(\frac{1}{z}))' + B(z)\overline{F}(\frac{1}{z})$ , that is, the part with non-negative powers of  $z$ , that we denote by  $D$ . Then we have

$$A(z)F'(z) + B(z)F(z) + D(z) + A(z)(\overline{F}(\frac{1}{z}))' + B(z)\overline{F}(\frac{1}{z}) - D(z) + 2H(z) = 0,$$

and therefore (12) and (13) hold.

Since  $(\overline{F}(\frac{1}{z}))' = -\frac{1}{z^2}\overline{F}'(\frac{1}{z})$ , we can write (13)

$$-\frac{1}{z^2}A(z)\overline{F}'(\frac{1}{z}) + B(z)\overline{F}(\frac{1}{z}) - D(z) = 0,$$

and therefore we have

$$-z^2\overline{A}(\frac{1}{z})F'(z) + \overline{B}(\frac{1}{z})F(z) - \overline{D}(\frac{1}{z}) = 0. \quad (15)$$

Thus, from (12) and (15), if we eliminate  $F'(z)$  we get

$$\left(z^2\overline{A}(\frac{1}{z})B(z) + A(z)\overline{B}(\frac{1}{z})\right)F(z) + \left(z^2\overline{A}(\frac{1}{z})(2H + D)(z) - A(z)\overline{D}(\frac{1}{z})\right) = 0.$$

Therefore

$$\begin{aligned} z^2\overline{A}(\frac{1}{z})B(z) + A(z)\overline{B}(\frac{1}{z}) &= 0 \text{ if and only if} \\ z^2\overline{A}(\frac{1}{z})(2H + D)(z) - A(z)\overline{D}(\frac{1}{z}) &= 0. \end{aligned}$$

Then one of the following conditions holds:

- (i) If  $z^2\overline{A}(\frac{1}{z})B(z) + A(z)\overline{B}(\frac{1}{z}) \neq 0$ , then  $F$  is rational.
- (ii) If  $z^2\overline{A}(\frac{1}{z})B(z) + A(z)\overline{B}(\frac{1}{z}) = 0$ , then

$$-\frac{A(z)}{z^2\overline{A}(\frac{1}{z})} = \frac{B(z)}{\overline{B}(\frac{1}{z})} = \frac{(2H + D)(z)}{-\overline{D}(\frac{1}{z})}.$$

On the other hand, if we conjugate coefficients in (12) and change  $z$  by  $\frac{1}{z}$  we can eliminate  $(\overline{F}(\frac{1}{z}))'$  with (13). Then, as before, either  $F$  is rational or (14) holds.

Now, if we put  $A(z) = z^\alpha \widehat{A}(z)$ ,  $B(z) = z^\beta \widehat{B}(z)$ ,  $(2H + D)(z) = z^\rho \widehat{(2H + D)}(z)$ , and  $D(z) = z^\sigma \widehat{D}(z)$ , relation (14) can be rewritten

$$-\frac{z^{p-2+\alpha}\widehat{A}}{A^*} = \frac{z^{q+\beta}\widehat{B}}{B^*} = \frac{z^{s+\rho}\widehat{(2H + D)}}{-D^*} = \frac{-z^{r+\sigma}\widehat{D}}{(2H + D)^*}.$$

Hence  $p + \alpha - 2 = q + \beta = s + \rho = r + \sigma$ , and since  $s - \sigma = r - \rho$ , we have  $\deg(\widehat{(2H + D)}) = \deg(\widehat{D})$ .

Next we study the relation

$$\frac{\widehat{(2H + D)}}{D^*} = \frac{\widehat{D}}{(2H + D)^*}.$$

If  $\widehat{(2H + D)}(z) = \prod_{j=1}^{r'_1}(z - \alpha_j)\prod_{j=1}^{r'_2}(z - \beta_j)$ , with  $|\alpha_j| = 1$ ,  $|\beta_j| \neq 1$ ,  $r'_1 + r'_2 = r - \rho$  and  $\widehat{D}(z) = \prod_{j=1}^{s'_1}(z - \gamma_j)\prod_{j=1}^{s'_2}(z - \delta_j)$ , with  $|\gamma_j| = 1$ ,  $|\delta_j| \neq 1$ ,  $s'_1 + s'_2 = s - \sigma$ , then the preceding equality is

$$\frac{\prod_{j=1}^{s'_1}\gamma_j\prod_{j=1}^{r'_1}(z - \alpha_j)\prod_{j=1}^{r'_2}(z - \beta_j)}{(-1)^{s'_1}\prod_{j=1}^{s'_1}(z - \gamma_j)\prod_{j=1}^{s'_2}(1 - \overline{\delta_j}z)} = \frac{\prod_{j=1}^{r'_1}\alpha_j\prod_{j=1}^{s'_1}(z - \gamma_j)\prod_{j=1}^{s'_2}(z - \delta_j)}{(-1)^{r'_1}\prod_{j=1}^{r'_1}(z - \alpha_j)\prod_{j=1}^{r'_2}(1 - \overline{\beta_j}z)}.$$

Hence  $r'_1 = s'_1$  and  $\alpha_j = \gamma_j \ \forall j$ . Therefore

$$\frac{\Pi_{j=1}^{r'_2}(z - \beta_j)}{\Pi_{j=1}^{s'_2}(1 - \bar{\beta}_j z)} = \frac{\Pi_{j=1}^{s'_2}(z - \delta_j)}{\Pi_{j=1}^{r'_2}(1 - \bar{\beta}_j z)}, \quad (16)$$

and since  $r - \rho = s - \sigma$ , then  $r'_2 = s'_2$ .

With respect to the zeros of modulus different from 1 we have the following possibilities:

- If there exists  $j_0$ ,  $1 \leq j_0 \leq r'_2$ , such that  $\beta_{j_0} = \delta_{j_0}$ , then  $\beta_{j_0}$  must be also a zero of  $A, B$  and  $H$ . Therefore the factor  $\frac{z - \beta_{j_0}}{1 - \bar{\beta}_{j_0} z}$  appears in  $\omega(z)$ . Let

$n_0$  ( $0 \leq n_0 \leq r'_2$ ) be the number of such zeros.

- If there exists  $j_0$ ,  $1 \leq j_0 \leq r'_2$ , such that  $\beta_{j_0} = \frac{1}{\delta_{j_0}}$ , then  $-\beta_{j_0}$  appears in the quotient  $\frac{\widehat{2H+D}}{D^*}$  and  $-\frac{1}{\beta_{j_0}}$  in  $\frac{\widehat{D}}{(2H+D)^*}$ . Since  $|\beta_{j_0}| \neq 1$ , the number  $n_1$  of zeros in this situation must be greater than 1.

Taking into account both cases (16) can be rewritten by

$$\prod_{j=1}^{n_0} \frac{(z - \beta_j)}{(1 - \bar{\beta}_j z)} \prod_{j=1}^{n_1} \frac{(z - \beta_j)}{(1 - \frac{z}{\bar{\beta}_j})} = \prod_{j=1}^{n_0} \frac{(z - \beta_j)}{(1 - \bar{\beta}_j z)} \prod_{j=1}^{n_1} \frac{(z - \frac{1}{\bar{\beta}_j})}{(1 - \bar{\beta}_j z)},$$

with  $n_0 + n_1 = r'_2$ .

Finally, since  $\prod_{j=1}^{n_1} \frac{(z - \beta_j)}{(1 - \frac{z}{\bar{\beta}_j})} = (-1)^{n_1} \Pi_{j=1}^{n_1} \beta_j$ , we obtain that

$$\omega(z) = (-1)^{r'_1+n_1} \Pi_{j=1}^{r'_1} \alpha_j \Pi_{j=1}^{n_1} \beta_j \prod_{j=1}^{n_0} \frac{(z - \beta_j)}{(1 - \bar{\beta}_j z)} z^k \text{ if } n_0 > 0,$$

and

$$\omega(z) = (-1)^{r'_1+r'_2} \Pi_{j=1}^{r'_1} \alpha_j \Pi_{j=1}^{n_1} \beta_j z^k \text{ if } n_0 = 0.$$

□

**Remark 1** Taking into account the preceding Theorem 3 and the fact that there exist Laguerre-Hahn affine functionals which are not semiclassical, we can conclude that if  $F$  satisfies (12) and (13) then this does not imply that  $\mathcal{L}$  is semiclassical. This gives a negative answer to the problem posed in [1]. Moreover, since a functional satisfying (12) and (13) is Laguerre-Hahn affine (see Theorem 4 below), Theorem 3 gives more information than Theorem 2 about the polynomial coefficients of equation (1).

Conversely, if we assume that the formal series  $F$  satisfies equation (12) then we can deduce some interesting results for  $G$ .

**Theorem 4** Let  $\mathcal{L}$  be a quasi definite and hermitian linear functional such that  $F$  satisfies

$$A(z)F'(z) + B(z)F(z) + R(z) = 0, \quad (17)$$

with  $A \neq 0$ .

(i) If  $A(z)\overline{B}(\frac{1}{z}) + z^2\overline{A}(\frac{1}{z})B(z) = 0$ , with  $B \neq 0$ , then  $\mathcal{L}$  is Laguerre-Hahn affine and the following relation holds

$$A(z)\overline{B}(\frac{1}{z})G'(z) + B(z)\overline{B}(\frac{1}{z})G(z) + \frac{1}{2}(R(z)\overline{B}(\frac{1}{z}) + B(z)\overline{R}(\frac{1}{z})) = 0.$$

(ii) If  $B(z) = 0$ , then  $\mathcal{L}$  is Laguerre-Hahn affine and

$$-z^2A(z)\overline{A}(\frac{1}{z})G'(z) + \frac{1}{2}(\overline{R}(\frac{1}{z})A(z) - z^2R(z)\overline{A}(\frac{1}{z})) = 0.$$

Proof: (i) If we take conjugates, change  $z$  by  $\frac{1}{z}$  in (17), and take into account that  $\overline{F'}(\frac{1}{z}) = -z^2(\overline{F}(\frac{1}{z}))'$ , we get

$$-z^2\overline{A}(\frac{1}{z})(\overline{F}(\frac{1}{z}))' + \overline{B}(\frac{1}{z})\overline{F}(\frac{1}{z}) + \overline{R}(\frac{1}{z}) = 0. \quad (18)$$

Now, from (18) and (17) we obtain that  $F$  satisfies the following equations

$$-z^2\overline{A}(\frac{1}{z})B(z)(\overline{F}(\frac{1}{z}))' + \overline{B}(\frac{1}{z})B(z)\overline{F}(\frac{1}{z}) + B(z)\overline{R}(\frac{1}{z}) = 0,$$

$$\overline{B}(\frac{1}{z})A(z)F'(z) + \overline{B}(\frac{1}{z})B(z)F(z) + \overline{B}(\frac{1}{z})R(z) = 0.$$

If  $A(z)\overline{B}(\frac{1}{z}) + z^2\overline{A}(\frac{1}{z})B(z) = 0$ , then

$$2A(z)\overline{B}(\frac{1}{z})G'(z) + 2B(z)\overline{B}(\frac{1}{z})G(z) + (R(z)\overline{B}(\frac{1}{z}) + B(z)\overline{R}(\frac{1}{z})) = 0.$$

(ii) If  $B = 0$ , proceeding in the same way, from (18) and (17), we get

$$-z^2\overline{A}(\frac{1}{z})A(z)(\overline{F}(\frac{1}{z}))' + A(z)\overline{R}(\frac{1}{z}) = 0,$$

$$-z^2\overline{A}(\frac{1}{z})A(z)F'(z) - z^2\overline{A}(\frac{1}{z})R(z) = 0.$$

Thus

$$-2z^2\overline{A}(\frac{1}{z})A(z)G'(z) + (\overline{R}(\frac{1}{z})A(z) - z^2\overline{A}(\frac{1}{z})R(z)) = 0.$$

□

**Corollary 2** *Let  $\mathcal{L}$  be a quasi-definite and hermitian functional such that*

$$A(z)F'(z) + B(z)F(z) + R(z) = 0, \text{ with } A \neq 0 \text{ and } B \neq 0.$$

(i) *If  $A(z)\overline{B}(\frac{1}{z}) + z^2\overline{A}(\frac{1}{z})B(z) = 0$ , then*

*$\mathcal{L}$  is semiclassical if and only if  $F$  is rational.*

(ii) *If  $A(z)\overline{B}(\frac{1}{z}) + z^2\overline{A}(\frac{1}{z})B(z) = 0$  and  $R(z)\overline{B}(\frac{1}{z}) + \overline{R}(\frac{1}{z})B(z) = 0$  then  $\mathcal{L}$  is semiclassical.*

Proof: (i) In this case we obtain, from Theorem 4, that  $\mathcal{L}$  satisfies the hypothesis of Theorem 2 and therefore

$\mathcal{L}$  is semiclassical if and only if  $G$  is rational.

Since  $G$  is rational if and only if  $F$  is rational, our result is an immediate consequence of this fact.

(ii) It is straightforward consequence of Theorem 4 (i).  $\square$

## 4 Transformations in the Laguerre-Hahn affine class

In the present section we consider some modifications of the moment functional that preserve the Laguerre-Hahn affine class.

In the following we assume that  $\mathcal{L}$  is a Laguerre-Hahn affine functional i.e. that the associated formal series  $G(z) = \sum_{n=-\infty}^{\infty} \overline{c}_n z^n$  satisfies equation (1). We will study some modifications of the functional such that the new functional remains a Laguerre-Hahn affine one.

(1) For  $\alpha \in \mathbb{C}$ , let us consider  $\tilde{\mathcal{L}} = |z - \alpha|^2 \mathcal{L}$ . It is easy to prove that  $\tilde{\mathcal{L}}$  is a hermitian functional and conditions on  $\mathcal{L}$  and  $\alpha$  in order to  $\tilde{\mathcal{L}}$  be quasi-definite, are very well-known (see [9]). When  $\tilde{\mathcal{L}}$  is quasi-definite we consider its formal series  $\tilde{G}(z) = \sum_{n=-\infty}^{\infty} \tilde{c}_n z^n$ , with  $\tilde{c}_n$  given by  $\tilde{c}_n = -\alpha c_{n-1} + (1 + |\alpha|^2)c_n - \overline{\alpha}c_{n+1}$ . Since  $z\tilde{G}(z) = [(1 + |\alpha|^2)z - \overline{\alpha}z^2 - \alpha]G(z)$  it follows that

$$\tilde{A}(z)\tilde{G}'(z) + \tilde{B}(z)\tilde{G}(z) + \tilde{H}(z) = 0, \text{ with}$$

$$\tilde{A}(z) = [(1 + |\alpha|^2)z - \overline{\alpha}z^2 - \alpha]z^2 A(z),$$

$$\tilde{B}(z) = \{[(1 + |\alpha|^2)z^2 - \overline{\alpha}z^3 - \alpha z]B(z) - (-\overline{\alpha}z^2 + \alpha)A(z)\}z, \text{ and}$$

$$\tilde{H}(z) = [(1 + |\alpha|^2)z - \overline{\alpha}z^2 - \alpha]^2 z H(z).$$

(2) Let  $\tilde{\mathcal{L}}$  be the moment functional such that  $|z-\alpha|^2\tilde{\mathcal{L}} = \mathcal{L}$ , with  $\alpha \in \mathbb{C}$ . The conditions to assure this functional  $\mathcal{L}$  is hermitian and quasi-definite are given in [5]. In this situation we consider the series  $\tilde{G}(z) = \sum_{n=-\infty}^{\infty} \tilde{c}_n z^n$ , with  $\tilde{c}_n$  given by  $-\alpha\tilde{c}_{n-1} + (1+|\alpha|^2)\tilde{c}_n - \bar{\alpha}\tilde{c}_{n+1} = c_n$ . Since  $zG(z) = [(1+|\alpha|^2)z - \bar{\alpha}z^2 - \alpha]\tilde{G}(z)$ , we obtain

$$\tilde{A}(z)\tilde{G}'(z) + \tilde{B}(z)\tilde{G}(z) + \tilde{H}(z) = 0, \text{ with}$$

$$\begin{aligned}\tilde{A}(z) &= [(1+|\alpha|^2)z - \bar{\alpha}z^2 - \alpha]zA(z), \\ \tilde{B}(z) &= [(1+|\alpha|^2)z - \bar{\alpha}z^2 - \alpha]zB(z) + (-\bar{\alpha}z^2 + \alpha)A(z), \text{ and} \\ \tilde{H}(z) &= z^2H(z).\end{aligned}$$

(3) Let  $\tilde{\mathcal{L}} = \mathcal{L} + \delta_\alpha$ , with  $\alpha \in \mathbb{C}$  and  $|\alpha| = 1$ .  $\tilde{\mathcal{L}}$  is hermitian and necessary and sufficient conditions for its regularity are given in [3].

We recall some well-known properties about derivatives of functionals. The derivative  $D\mathcal{L}$  of a linear hermitian functional  $\mathcal{L}$  is defined by  $D\mathcal{L}(P(z)) = -i\mathcal{L}(zP'(z))$ . This definition is motivated by the differential behavior of positive measures on the unit circle with respect to the integration on the unit circle, (see [1]). It is easy to prove that  $D(A(z)\mathcal{L}) = A(z)D\mathcal{L} + izA'(z)\mathcal{L}$ .

Furthermore Laguerre-Hahn affine functionals are characterized as follows :  
 $G$  satisfies equation (1)

$$A(z)G'(z) + B(z)G(z) + H(z) = 0,$$

if and only if  $\mathcal{L}$  satisfies

$$-iD(A(z)\mathcal{L}) + z(B(z) - A'(z))\mathcal{L} + zH(z)L_1 = 0, \quad (19)$$

where  $L_1$  is the linear functional corresponding to the Lebesgue normalized measure.

When  $\tilde{\mathcal{L}}$  is quasi-definite we consider the series  $\tilde{G}(z) = \sum_{n=-\infty}^{\infty} (c_n + \bar{\alpha}^n)z^n = G(z) + \sum_{n=-\infty}^{\infty} \bar{\alpha}^n z^n$ .

Taking into account the definition of  $\delta_\alpha$  we have

$$D[(z-\alpha)^k A(z)\delta_\alpha](z^n) = 0, \quad \forall n \in \mathbb{Z}, \quad \forall k \geq 1,$$

and

$$[(z-\alpha)^k B(z)\delta_\alpha](z^n) = 0 \quad \forall n \in \mathbb{Z}, \quad \forall k \geq 1.$$

Now we compute

$$\begin{aligned}D[(z-\alpha)^2 A(z)\tilde{\mathcal{L}}](z^n) &= D[(z-\alpha)^2 A(z)(\mathcal{L} + \delta_\alpha)](z^n) = \\ D[(z-\alpha)^2 A(z)\mathcal{L}](z^n) &= (z-\alpha)^2 D(A(z)\mathcal{L})(z^n) + 2iz(z-\alpha)A(z)\mathcal{L}(z^n).\end{aligned}$$

Since  $\mathcal{L}$  satisfies (19) then

$$D[(z - \alpha)^2 A(z) \tilde{\mathcal{L}}](z^n) = -iz(z - \alpha)[(z - \alpha)(B(z) - A'(z)) - 2A(z)] \tilde{\mathcal{L}}(z^n) - iz(z - \alpha)^2 H(z) L_1(z^n).$$

Thus  $\tilde{\mathcal{L}}$  is Laguerre-Hahn affine and for  $\tilde{G}$  we get

$$(z - \alpha)^2 A(z) \tilde{G}'(z) + (z - \alpha)^2 B(z) \tilde{G}(z) + (z - \alpha)^2 H(z) = 0.$$

(4) Let  $\tilde{\mathcal{L}} = \mathcal{L} + L_k$ , with  $L_k$  a regular and hermitian functional such that its formal series  $G_k$  is given by  $G_k(z) = \sum_{n=-k}^k \overline{b_n} z^n$ . Since there exist polynomials  $A_k$  and  $B_k$  such that  $A_k(z) G'_k(z) + B_k(z) G_k(z) = 0$ , if  $\tilde{\mathcal{L}}$  is quasi-definite then  $\tilde{G} = G + G_k$  satisfies the following equation

$$z^k A_k(z) A(z) \tilde{G}'(z) + z^k A_k(z) B(z) \tilde{G}(z) + z^k [(AB_k - A_k B)(z) + (A_k H)(z)] = 0.$$

Therefore  $\tilde{\mathcal{L}}$  is Laguerre-Hahn affine.

(5) Let  $\tilde{\mathcal{L}}$  be the hermitian functional with moments  $\tilde{c}_n$  given by  $\tilde{c}_n = e^{in\varphi} c_n$ ,  $\varphi \in [0, 2\pi]$ . Assuming  $\tilde{\mathcal{L}}$  quasi-definite we consider the series  $\tilde{G}(z) = \sum_{n=-\infty}^{\infty} \tilde{c}_n z^n$ . Since  $\tilde{G}(z) = G(e^{-i\varphi} z)$ , then

$$\tilde{A}(z) \tilde{G}'(z) + \tilde{B}(z) \tilde{G}(z) + \tilde{H}(z) = 0, \text{ with}$$

$$\tilde{A}(z) = A(e^{-i\varphi} z), \tilde{B}(z) = e^{-i\varphi} B(e^{-i\varphi} z) \text{ and } \tilde{H}(z) = e^{-i\varphi} H(e^{-i\varphi} z).$$

(6) Let  $\tilde{\mathcal{L}}$  be the functional with moments  $\tilde{c}_n$  given by  $\tilde{c}_n = \overline{c_n}$ .  $\tilde{\mathcal{L}}$  is hermitian and quasi-definite. Taking into account that  $\tilde{G}(z) = G(\frac{1}{z})$ , then  $\tilde{G}(z)$  satisfies

$$\tilde{A}(z) \tilde{G}'(z) + \tilde{B}(z) \tilde{G}(z) + \tilde{H}(z) = 0,$$

with  $\tilde{A}(z) = z^{t+2} A(\frac{1}{z})$ ,  $\tilde{B}(z) = -z^t B(\frac{1}{z})$ , and  $\tilde{H}(z) = -z^t H(\frac{1}{z})$ , where  $t = \max\{\deg(A), \deg(B), \deg(H)\}$ .

(7) Let  $\tilde{\mathcal{L}}$  be the hermitian functional with moments  $\tilde{c}_{nh} = c_n$  and  $\tilde{c}_{nh+j} = 0$  for  $j = 1, \dots, h-1$ . If  $\tilde{\mathcal{L}}$  is quasi-definite consider the associated series  $\tilde{G}$ . Since  $\tilde{G}(z) = G(z^h)$  we get

$$\tilde{A}(z) \tilde{G}'(z) + \tilde{B}(z) \tilde{G}(z) + \tilde{H}(z) = 0, \text{ with}$$

$$\tilde{A}(z) = A(z^h), \tilde{B}(z) = h z^{h-1} B(z^h), \tilde{H}(z) = h z^{h-1} H(z^h).$$



**Remark 2** *In the semiclassical case the modifications (1), (2) and (3) were studied in [14]. Modifications (5), (6) and (7) have not been studied in the semiclassical case but it is clear that they also preserve the semiclassical character. Finally, modification (4) does not preserve the semiclassical character in general.*

## 5 Differential properties

In ([14]) it was proved that the monic orthogonal polynomial of degree  $n$  with respect to a semiclassical functional satisfies an homogeneous second order linear differential equation with polynomial coefficients of bounded degree. Furthermore, in [6] it was proved, in the semiclassical case, that the reversed of the monic orthogonal polynomial of degree  $n$  satisfies an homogeneous second order linear differential equation with coefficients depending on  $n$ .

Now for Laguerre-Hahn affine functionals in the Szegő's class we deduce that the reversed of the monic orthogonal polynomial of degree  $n$  satisfies a Riccati differential equation with coefficients depending on  $n$ .

Let  $\mathcal{L}$  be a Laguerre-Hahn affine functional, i.e.  $G$  satisfies (1). Let us consider the monic orthogonal polynomial sequence  $\{\Phi_n\}$  with respect to  $\mathcal{L}$ , i.e.,

$$\mathcal{L}(\Phi_n(z)z^{-k}) = \begin{cases} 0 & k = 0, \dots, n-1 \\ e_n \neq 0 & k = n. \end{cases}$$

If  $\{\Psi_n\}$  is the sequence of polynomials of the second kind with respect to  $\mathcal{L}$  (see [8]), it is well-known that  $\{\Psi_n\}$  is orthogonal with respect to a functional  $\mathcal{L}$  with formal series  $F_\Psi = \frac{1}{F}$ . Then, since  $F$  satisfies the differential equation (12), it is clear that the series  $F_\Psi$  satisfies the following Bernoulli's differential equation:

$$-A(z)F'_\Psi(z) + B(z)F_\Psi(z) + (2H(z) + D(z))(F_\Psi(z))^2 = 0. \quad (20)$$

If we assume that  $\mathcal{L}$  is a positive definite functional in the Szegő's class, then  $\mathcal{L}$  is a positive definite functional in the Szegő's class (see [8]). In this situation

$$\lim_{n \rightarrow \infty} \frac{\Phi_n^*(z)}{\Psi_n^*(z)} = F_\Psi(z) \text{ uniformly on compact subsets of } |z| < 1,$$

$$\lim_{n \rightarrow \infty} \Phi_n^*(z) = \Pi(z) \text{ uniformly on compact subsets of } |z| < 1,$$

and

$$\lim_{n \rightarrow \infty} \Psi_n^*(z) = \Pi_\Psi(z) \text{ uniformly on compact subsets of } |z| < 1,$$

with  $\Pi$  and  $\Pi_\Psi$  the Szegő functions corresponding to the functionals  $\mathcal{L}$  and  $\mathcal{L}_\Psi$ . We recall that the Szegő function  $\Pi$  is defined by

$$\Pi(z) = \exp \left( -\frac{1}{4\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log \mu'(\theta) d\theta \right) \exp \left( \frac{1}{4\pi} \int_0^{2\pi} \log \mu'(\theta) d\theta \right).$$

Then we prove the following result.

**Theorem 5** *Let  $\mathcal{L}$  be a positive definite functional in the Szegő's class with Szegő's function  $\Pi$ . If  $\mathcal{L}$  is Laguerre-Hahn affine and  $F$  satisfies  $A(z)F'(z) + B(z)F(z) + 2H(z) + D(z) = 0$  then, for each  $n$ , the reversed polynomial  $\Phi_n^*$  satisfies the following Riccati differential equation*

$$-A(z)F(z)\Pi(z)(\Phi_n^*(z))' + [A(z)F(z)\Pi'(z) - (2H(z) + D(z))\Pi(z)]\Phi_n^*(z) + (2H(z) + D(z))(\Phi_n^*(z))^2 = Z_n(z),$$

where  $Z_n$  is an analytic function in the unit disk and

$$\lim_{n \rightarrow \infty} Z_n(z) = 0, \text{ uniformly on compact subsets of } |z| < 1.$$

Proof: Since  $F_\Psi = \frac{\Pi}{\Pi_\Psi}$ , if we substitute in (20), we get

$$-A(z)[\Pi(z)\Pi'_\Psi(z) - \Pi'(z)\Pi_\Psi(z)] + B(z)\Pi(z)\Pi_\Psi(z) + (2H + D)(z)\Pi^2(z) = 0,$$

and therefore, since that  $\lim_{n \rightarrow \infty} \Phi_n^*(z) = \Pi(z)$ ,

$$\lim_{n \rightarrow \infty} (A(z)\Pi'_\Psi(z)\Phi_n^*(z) - A(z)\Pi_\Psi(z)(\Phi_n^*(z))' + B(z)\Pi_\Psi(z)\Phi_n^*(z) + (2H(z) + D(z))(\Phi_n^*(z))^2) = 0,$$

uniformly on compact subsets of  $|z| < 1$ .

Again, taking into account that  $\Pi_\Psi = F\Pi$ ,

$$\lim_{n \rightarrow \infty} (-A(z)F(z)\Pi(z)(\Phi_n^*(z))' + [A(z)(F\Pi)'(z) + B(z)F(z)\Pi(z)]\Phi_n^*(z) + (2H(z) + D(z))(\Phi_n^*(z))^2) = 0.$$

Then, using (12), we obtain for each  $n$ , the following differential equation

$$-A(z)F(z)\Pi(z)(\Phi_n^*(z))' + [A(z)F(z)\Pi'(z) - (2H(z) + D(z))\Pi(z)]\Phi_n^*(z) + (2H(z) + D(z))(\Phi_n^*(z))^2 = Z_n(z),$$

with  $Z_n(z) \in H(D)$  and such that  $\lim_{n \rightarrow \infty} Z_n(z) = 0$ , uniformly on compact subsets of  $|z| < 1$ .  $\square$

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# Convolution of Fuzzy Multifunctions and Applications

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**Abstract**—The purpose of this article is to define intersection convolution of fuzzy multifunctions defined between groups and study their properties. As application we obtained extension of linear selector fuzzy multifunction.

**KEYWORDS AND PHRASES** : Fuzzy multifunction, convolution, selector, extension, group, vector space.

**2000 MATHEMATICS SUBJECT CLASSIFICATIONS**: 47S40, 46S40, 44A35, 47H04, 03E72, 54C65

## 1. INTRODUCTION

Convolutions of integrable functions are of fundamental importance and have several interesting applications in applied mathematics and probability theory (see Balakrishnan [4] and Khan [10] ). In last thirty years the theory of multifunctions has advanced in a variety of ways. The theory of multifunctions was first codified by Berge [7]. Applications of this theory can be found in economic theory, noncooperative games, artificial intelligence, medicine and existence of solutions for differential inclusions ( see Klein and Thompson[11], Aliprantis and Border [2] and references therein). Arens [3], Nikodem [12], Smajdor [13] and Stromberg [14] have initiated the study of calculus of multifunctions/relations and obtained many beautiful significant results in this area.

Recently Heilpern [9], Butnariu [8], Albrycht and Maltoka [1], Tsiporkova-Hristoskova, De Baets and Kerre[15] and Beg [5,6] have started the study of fuzzy multifunctions and investigated different fundamental properties. The aim of this article is to define the intersection convolution of fuzzy multifunctions defined from a group  $X$  into another group  $Y$ , and then to study their basic properties. As application we prove existence of an extension of linear selector fuzzy multifunction from a vector subspace under suitable conditions.

## 2. PRELIMINARIES

Let  $X$  be an arbitrary (nonempty) set. A *fuzzy set* (in  $X$ ) is a function with domain  $X$  and values in  $[0, 1]$ . If  $A$  is a fuzzy set and  $x \in X$ , the function value  $A(x)$  is called the *grade of membership* of  $x$  in  $A$ . Let  $A$  and  $B$  be fuzzy sets in  $X$ . We write  $A \subseteq B$  if  $A(x) \leq B(x)$  for each  $x \in X$ . For any family  $\{A_i\}_{i \in I}$  of fuzzy sets in  $X$ , we define  $[\bigcap_{i \in I} A_i](x) = \inf_{i \in I} A_i(x)$ .

**Definition 2.1.** A *fuzzy multifunction*  $F$  from a set  $X$  into a set  $Y$  assigns to each  $x$  in  $X$  a fuzzy subset  $F(x)$  of  $Y$ . We denote it by  $F : X \rightsquigarrow Y$ . We can also identify  $F$  with a fuzzy subset  $F$  of  $X \times Y$  with  $F(x, y) = [F(x)](y)$ .

If  $A$  is a fuzzy subset of  $X$ , then the fuzzy set  $F(A)$  in  $Y$  is defined by

$$[F(A)](y) = \sup_{x \in X} [F(x, y) \wedge A(x)].$$

The set  $D_F = \{x \in X : F(x) \text{ is nonempty}\}$  is called the *domain* of  $F$ . If  $D_F = X$ , then  $F$  is called a fuzzy multifunction from  $X$  into  $Y$ . If  $\text{Domain}(F) \neq X$ , then  $F$  is called a fuzzy multifunction between  $X$  and  $Y$ .

Let  $F$  and  $T$  be fuzzy multifunctions from  $X$  into  $Y$  such that  $T \subset F$ , or equivalently  $T(x) \subset F(x)$  for all  $x \in X$ , then  $T$  is a *selector fuzzy multifunction* of  $F$ . The selector fuzzy multifunction of the restriction  $F|_Z = F \cap (Z \times Y)$  of  $F$  to a subset  $Z$  of  $X$  is called *partial fuzzy selector* of  $F$ .

Let  $X$  be an additive group (with " + " as binary operation) and  $A, B$  fuzzy subsets of  $X$  then,

$$A + B = \{x + y : x \in A, y \in B\},$$

with ,

$$(A + B)(z) = \sup_{z=x+y} \inf\{A(x), B(y)\}.$$

Also,

$$-A = \{-x : A(x) \neq 0\},$$

with,

$$-A(-x) = A(x),$$

and,

$$A - B = A + (-B).$$

If  $X$  is a vector space (over  $\Gamma$ ) then we write  $\lambda A = \{\lambda x : A(x) \neq 0\}$  for all  $\lambda \in \Gamma$  with  $\lambda A(\lambda x) = A(x)$ .

A fuzzy multifunction  $F$  between groups  $X$  and  $Y$  is called *superadditive* (*resp. subadditive*) if,

$$F(x) + F(y) \subset F(x + y) \text{ (resp. } F(x + y) \subset F(x) + F(y)),$$

for all  $x, y$  in  $X$ , and *additive* if it is both superadditive and subadditive. Moreover,  $F$  is said to be *odd* if  $F(-x) = -F(x)$  for all  $x$  in  $X$ .

**Remark 2.2.** (i)  $F$  is odd if and only if  $-F(x) \subset F(-x)$  (or equivalently  $F(-x) \subset -F(x)$  for all  $x \in X$ ). (ii) An odd superadditive fuzzy multifunction between groups is additive.

A fuzzy multifunction  $F$  between vector spaces  $X$  and  $Y$  (over  $\Gamma$ ) is called *homogeneous* if  $F(\lambda x) = \lambda F(x)$  for all nonzero  $\lambda \in \Gamma$  and  $x \in X$ . The fuzzy multifunction  $F$  is homogeneous if and only if  $\lambda F(x) \subset F(\lambda x)$  (or equivalently  $F(\lambda x) \subset \lambda F(x)$ ) for all nonzero  $\lambda \in \Gamma$  and  $x \in X$ . A homogeneous

fuzzy multifunction  $F$  is called *linear* if it is additive. Homogeneous fuzzy multifunctions are odd.

For more details we refer to [5,6,15,16].

### 3. INTERSECTION CONVOLUTION

**Definition 3.1.** Let  $X$  and  $Y$  be two groups and  $Z$  be a subgroup of  $X$ . If  $F : X \rightsquigarrow Y$  is a fuzzy multifunction and  $T : Z \rightsquigarrow Y$  then the fuzzy multifunction  $F * T$  defined by,

$$(F * T)(x) = \bigcap_{z \in Z} [F(x - z) + T(z)],$$

for all  $x \in X$ , is called the *intersection convolution* of  $F$  and  $T$ .

The fuzzy multifunction  $[F + T(0)] : X \rightsquigarrow Y$  is defined by,

$$(F + T(0))(x) = F(x) + T(0).$$

Similarly,  $[F(0) + T] : Z \rightsquigarrow Y$  is defined by

$$(F(0) + T)(z) = F(0) + T(z).$$

**Theorem 3.2.** (i)  $F * T \subset F + T(0)$  and (ii).  $(F * T)|_Z \subset F(0) + T$ .

**Proof.** By definition 3.1,

$$(F * T)(x) \subset F(x) + T(0) = (F + T(0))(x),$$

for all  $x \in X$ .

And,

$$(F * T)(z) \subset F(z - z) + T(z) = (F(0) + T)(z),$$

for all  $z \in Z$ .

**Corollary 3.3.** (i). If

$$[T(0)](y) = \begin{cases} 0, & \text{if } y \neq 0, \\ r, & \text{if } y = 0, \end{cases} \quad \text{where } r \in (0, 1],$$



then  $(F * T) \subset T$ , and

(ii). If

$$[F(0)](y) = \begin{cases} 0, & \text{if } y \neq 0, \\ r, & \text{if } y = 0, \end{cases} \quad \text{where } r \in (0, 1],$$

then  $(F * T)|_Z \subset T$ .

**Theorem 3.4.** Let  $X$  and  $Y$  be two groups and  $Z$  be a subgroup of  $X$ . If  $F : X \rightsquigarrow Y$  is a fuzzy multifunction and  $T$  is a subadditive selector fuzzy multifunction of  $F|_Z$ , then  $T \subset (F * T)|_Z$ .

Moreover, if

$$[F(0)](y) = \begin{cases} 0, & \text{if } y \neq 0, \\ r, & \text{if } y = 0, \end{cases} \quad \text{where } r \in (0, 1],$$

then  $T = (F * T)|_Z$ .

**Proof.** If  $z \in Z$ , then

$$\begin{aligned} T(z) &= T(z - v + v) \subset T(z - v) + T(z) \\ &\subset F(z - v) + T(z) \end{aligned}$$

for all  $v \in Z$ . Hence  $T \subset (F * T)|_Z$ .

If

$$[F(0)](y) = \begin{cases} 0, & \text{if } y \neq 0, \\ r, & \text{if } y = 0, \end{cases} \quad \text{where } r \in (0, 1],$$

then by corollary 3.3 the converse inclusion also hold. Hence  $T = (F * T)|_Z$ .

**Corollary 3.5.** If  $F$  is a fuzzy multifunction such that,

$$[F(0)](y) = \begin{cases} 0, & \text{if } y \neq 0, \\ r, & \text{if } y = 0, \end{cases} \quad \text{where } r \in (0, 1],$$

and  $T$  is a subadditive selector fuzzy multifunction of  $F$  then  $T = F * T$ .

**Corollary 3.6.** If  $F$  is a subadditive fuzzy multifunction such that

$$[F(0)](y) = \begin{cases} 0, & \text{if } y \neq 0, \\ r, & \text{if } y = 0, \end{cases} \quad \text{where } r \in (0, 1],$$

then  $F = F * F$ .

**Theorem 3.7.** Let  $X$  and  $Y$  be two groups and  $Z$  be a subgroup of  $X$ . If  $F : X \rightsquigarrow Y$  is a fuzzy multifunction and  $T$  is an additive selector fuzzy multifunction of  $F|_Z$ , then  $T$  is also a selector fuzzy multifunction of  $(F + T(0))|_Z$  and

$$F * T = (F + T(0)) * T.$$

**Proof.** For  $z \in Z$ ,

$$T(z) = T(z) + T(0) \subset F(z) + T(0) = (F + T(0))(z).$$

Therefore  $T$  is a selector fuzzy multifunction of  $(F + T(0))|_Z$ . For  $x \in X$ ,

$$\begin{aligned} (F * T)(x) &= \bigcap_{z \in Z} (F(x - z) + T(z)) \\ &= \bigcap_{z \in Z} (F(x - z) + T(0) + T(z)) \\ &= \bigcap_{z \in Z} ((F + T(0))(x - z) + T(z)) \\ &= ((F + T(0)) * T)(x) \end{aligned}$$

Hence  $F * T = (F + T(0)) * T$ .

**Theorem 3.8.** Let  $X$  and  $Y$  be two groups and  $Z$  be a subgroup of  $X$ . Let  $F : X \rightsquigarrow Y$  be a fuzzy multifunction,  $T$  a subadditive selector fuzzy multifunction of  $F|_Z$  and  $S$  is a subadditive selector fuzzy multifunction of  $F$  such that  $T = S|_Z$ , then  $S \subset F * T$ .

**Proof.** Theorem 3.4 and definition 3.1 imply

$$\begin{aligned} S(x) \subset (F * S)(x) &= \bigcap_{z \in X} (F(x - z) + S(z)) \subset \bigcap_{z \in Z} (F(x - z) + S(z)) \\ &= \bigcap_{z \in Z} (F(x - z) + T(z)) = (F * T)(x) \end{aligned}$$

for all  $x \in X$ . Hence  $S \subset F * T$ .

**Corollary 3.9.** If  $F$  and  $T$  are as in theorem 3.8 and  $T$  can be extended to a subadditive selector fuzzy multifunction of  $F$  then  $(F * T)(x)$  is nonempty for all  $x \in X$ .

**Proof.** Assume that  $T$  can be extended to  $S$ , then  $S(x) \subset (F * T)(x)$ . Hence  $(F * T)(x)$  is nonempty for all  $x$  in  $X$ .

**Corollary 3.10.** If  $F$  is subadditive fuzzy multifunction with

$$[F(0)](y) = \begin{cases} 0, & \text{if } y \neq 0, \\ r, & \text{if } y = 0, \end{cases} \quad \text{where } r \in (0, 1],$$

then  $F = F * (F|_Z)$ .

**Theorem 3.11.** Let  $X$  and  $Y$  be two groups,  $Z$  be a subgroup of  $X$  and  $F : X \rightsquigarrow Y$  a fuzzy multifunction. If  $T$  is an additive selector fuzzy multifunction of  $F|_Z$ , which can be extended to a subadditive selector fuzzy multifunction of  $F + T(0)$  then  $(F * T)(x)$  is nonempty for all  $x \in X$ .

**Proof.** Theorem 3.7 implies that  $T$  is a selector fuzzy multifunction of  $(F + T(0))|_Z$ . Theorem 3.7 and corollary 3.9 further imply that  $(F * T)(x) = ((F + T(0)) * T)(x) \neq \phi$ , for all  $x \in X$ .

**Theorem 3.12.** Let  $X$  and  $Y$  be two groups  $Z$  be a subgroup of  $X$  and  $F : X \rightsquigarrow Y$  be a superadditive fuzzy multifunction. If  $T$  is a selector fuzzy multifunction of  $F|_Z$  with  $0 \in T(0)$ , then  $F = F + T(0)$ .

**Proof.** For  $x \in X$ ,

$$F(x) \subset F(x) + T(0) = (F + T(0))(x),$$

and

$$(F + T(0))(x) = F(x) + T(0) \subset F(x) + F(0) \subset F(x).$$

It implies that,  $F = F + T(0)$ .

**Theorem 3.13** Let  $X$  and  $Y$  be two group,  $Z$  be a subgroup of  $X$  and  $F : X \rightsquigarrow Y$  be a superadditive fuzzy multifunction. If  $T$  is a subadditive

selector fuzzy multifunction of  $F|_Z$  then  $F * T = F + T(0)$ .

**Proof.** For  $x \in X$ ,

$$\begin{aligned} (F + T(0))(x) &= F(x) + T(0) \subset F(x) + T(-z) + T(z) \\ &\subset F(x) + F(-z) + T(z) \subset F(x - z) + T(z), \end{aligned}$$

for all  $z \in Z$ . It implies that,

$$(F + T(0))(x) \subset (F * T)(x).$$

Theorem 3.2 further implies  $(F * T)(x) \subset (F + T(0))(x)$ . Hence  $F * T = F + T(0)$ .

**Remark 3.14.**(i). If  $F$  and  $T$  are as in theorem 3.13, and  $0 \in T(0)$ , then  $F = F * T$

(ii) If  $F$  is an additive fuzzy multifunction from a group  $X$  into another group  $Y$  with  $0 \in F(0)$  and  $Z$  is a subgroup of  $X$ , then  $F = F * (F|_Z)$ .

(iii) If  $F$  is an additive fuzzy multifunction from a group  $X$  into another group  $Y$  with  $0 \in F(0)$ , then  $F = F * F$ .

**Theorem 3.15.** Let  $X$  and  $Y$  be two groups and  $Z$  be a subgroup of  $X$ . Let  $F : X \rightsquigarrow Y$  be a fuzzy multifunction and  $T : Z \rightsquigarrow Y$  be an odd additive fuzzy multifunction then

$$(F * T)(x + z) = (F * T)(x) + T(z),$$

for all  $x \in X$  and  $z \in Z$ .

**Proof.** Let  $x \in X$  and  $z \in Z$ , then for  $u \in Z$ ,

$$\begin{aligned} (F * T)(x) + T(z) &\subset F(x - (u - z)) + T(u - z) + T(z) \\ &= F((x + z) - u) + T(u) = (F * T)(x + z). \end{aligned}$$

It further implies that

$$(F * T)(x + z) + T(-z) \subset (F * T)(x).$$

Also

$$(F * T)(x + z) \subset (F * T)(x) - T(-z) = (F * T)(x) + T(z).$$

Hence

$$(F * T)(x + z) = (F * T)(x) + T(z).$$

**Corollary 3.16.** Let  $X$  and  $Y$  be two groups,  $Z$  be a subgroup of  $X$ , and  $F : X \rightsquigarrow Y$  be a fuzzy multifunction with

$$[F(0)](y) = \begin{cases} 0, & \text{if } y \neq 0, \\ r, & \text{if } y = 0, \end{cases} \quad \text{where } r \in (0, 1].$$

If  $T$  is an additive selector fuzzy multifunction of  $F|_Z$ , then

$$(F * T)(x + z) = (F * T)(x) + (F * T)(z)$$

for all  $x \in X$  and  $z \in Z$ .

**Proof.** Using theorem 3.15 and theorem 3.4, we have

$$\begin{aligned} (F * T)(x + z) &= (F * T)(x) + T(z) \\ &= (F * T)(x) + (F * T)(z) \end{aligned}$$

for all  $x \in X$  and  $z \in Z$ .

#### 4. APPLICATION TO EXTENSION PROBLEM

**Theorem 4.1.** Let  $X$  and  $Y$  be two vector spaces (over  $\Gamma$ ) and  $Z$  be a subspace of  $X$ . If  $F : X \rightsquigarrow Y$  is a homogeneous fuzzy multifunction and  $T : Z \rightsquigarrow Y$  is a homogeneous fuzzy multifunction then  $F * T$  is also homogeneous.

**Proof.** Let  $\lambda$  be any nonzero element of  $\Gamma$  and  $x \in X$ , then for all  $z \in Z$ ,

$$\begin{aligned} \lambda(F * T)(x) &\subset \lambda \left( F\left(x - \frac{1}{\lambda}z\right) + T\left(\frac{1}{\lambda}z\right) \right) \\ &= \lambda F\left(x - \frac{1}{\lambda}z\right) + \lambda T\left(\frac{1}{\lambda}z\right) \\ &= F(\lambda x - z) + T(z) \\ &= (F * T)(\lambda x). \end{aligned}$$

Therefore  $\lambda(F * T)(x) \subset (F * T)(\lambda x)$ . Hence  $F * T$  is homogeneous.

**Theorem 4.2.** Let  $X$  and  $Y$  be two vector spaces (over  $\Gamma$ ) and  $Z$  a vector subspace of  $X$  such that  $\text{codim}(Z) = 1$ . Let  $F : X \rightsquigarrow Y$  be a homogeneous fuzzy multifunction and  $T$  be a linear selector fuzzy multifunction of  $F|_Z$ , then: there exists a linear selector fuzzy multifunction  $S$  of  $F + T(0)$  such that  $S|_Z = T$  if and only if  $(F * T)(x)$  is nonempty for some  $x \in X \setminus Z$ .

**Proof.** Assume that there exists a linear selector fuzzy multifunction  $S$  of  $F + T(0)$  such that  $S|_Z = T$ . Theorem 3.11 implies  $(F * T)(x)$  is nonempty for all  $x \in X$ .

Conversely, let there exists  $x_0 \in X \setminus Z$  such that  $(F * T)(x_0)$  is nonempty. It implies that there exists some  $y_0 \in Y$  such that  $y_0 \in (F * T)(x_0)$ . Since  $\text{codim}(Z) = 1$ , therefore  $\text{span}\{x_0\} \oplus Z = X$ . Define  $S : X \rightsquigarrow Y$  by

$$S(\lambda x_0 + z) = \lambda y_0 + T(z)$$

for all  $\lambda \in \Gamma$  and  $z \in Z$ .

Obviously,  $S$  is well defined linear fuzzy multifunction from  $X$  into  $Y$  and  $S|_Z = T$ . For  $z \in Z$ , we have

$$\begin{aligned} S(\lambda x_0 + z) &= \lambda y_0 + T(z) \\ &\subset \lambda(F * T)(x_0) + T(z) \\ &\quad (\text{By Theorem 4.1}) \\ &\subset (F * T)(\lambda x_0) + T(z) \\ &\quad (\text{By Theorem 3.15}) \\ &= (F * T)(\lambda x_0 + z) \\ &\quad (\text{By Theorem 3.2}) \\ &\subset F(\lambda x_0 + z) + T(0) \\ &= (F + T(0))(\lambda x_0 + z) \end{aligned}$$

for all non-zero  $\lambda$  in  $\Gamma$ . Also for  $z \in Z$ , using theorem 3.4 and theorem 3.2. we obtain,

$$S(z) = T(z) \subset (F * T)(z) \subset F(z) + T(0) = (F + T(0))(z).$$

It further implies that  $S$  is a linear selector fuzzy multifunction of  $F + T(0)$  with  $S|_Z = T$ .

**Theorem 4.3.** Let  $X$  and  $Y$  be two vector spaces (over  $\Gamma$ ) and  $Z$  be a subspace of  $X$ . Let  $F : X \rightsquigarrow Y$  be a homogeneous fuzzy multifunction,  $T$  be a linear selector fuzzy multifunction of  $F|_Z$  and  $G \subset F + T(0)$  be a linear fuzzy multifunction with  $T = G|_Z$ . If  $(F * G)(x)$  is nonempty for all  $x \notin D_G$  then there exists a linear selector fuzzy multifunction  $S$  of  $F + T(0)$  with  $T = S|_Z$ .

**Proof.** Let  $\mathcal{L}$  be the family of all linear fuzzy multifunction  $G \subset F + T(0)$  with  $T = G|_Z$ . Since  $T$  is a linear selector fuzzy multifunction of  $F + T(0)$  (see theorem 3.7) therefore  $T \in \mathcal{L}$ . The family  $\mathcal{L}$  is partially ordered by inclusion. By the Hausdorff maximality principle, there exists a maximal totally ordered subset  $\mathcal{B}$  of  $\mathcal{L}$ .

Define  $S = \cup \mathcal{B}$ . If  $G_1, G_2 \in \mathcal{B}$ , then either  $G_1 \subset G_2$  or  $G_2 \subset G_1$ . It further implies  $S$  is a linear fuzzy multifunction from some subspace  $W$  of  $X$  into  $Y$ . Since for any  $G \in \mathcal{B}$ ,  $T = G|_Z$  and  $G \subset F + T(0)$ , therefore  $T = S|_Z$  and  $S \subset F + T(0)$ .

We claim  $W = X$ . Assume that, on the contrary  $W \neq X$ , then there exists an  $x_0 \in X$  such that  $x_0 \notin W$ .

For  $x_0 \in X \setminus W$ ,

$$((F + T(0)) * S)(x_0) = ((F + S(0)) * S)(x_0)$$

(By Theorem 3.7)

$$= (F * S)(x_0)$$

(By hypothesis)

$$\neq \phi.$$

Theorem 4.2 further implies that there exists a linear selector fuzzy multifunction  $P$  of  $(F + T(0)) + S(0)$  to the subspace  $(\text{span}\{x_0\}) \oplus W$  of  $X$  with  $S = P|_W$  and thus  $T = P|_Z$ .

But  $\mathcal{B} \cup \{P\}$  is a totally ordered subset of  $\mathcal{L}$ , contradiction to the maximality of  $\mathcal{B}$ , because  $P \notin B$ . Hence  $W = X$ .

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# On the solutions of differential equation $\epsilon^2 (a^2(t)y')' + p(t)f(y) = 0$ with arbitrarily large zero number

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## Abstract

We consider initial problem for a singularly perturbed differential equation  $\epsilon^2 (a^2(t)y')' + p(t)f(y) = 0$  on a finite interval  $[0, \tau]$ , where  $f$  is an S-shaped curve with three simple zeros and sublinear growth at infinity. We find that there exists a decreasing sequence  $\{\epsilon_n\}_{n=n_0}^\infty$ ,  $\lim_{n \rightarrow \infty} \epsilon_n = 0$  such that the corresponding solution has  $n$  zeros on  $(0, \tau)$  and  $y(\tau, \epsilon_n) = 0$ .

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## 1 Introduction

We consider the second order singularly perturbed differential equation

$$\epsilon^2 (a^2(t)y')' + p(t)f(y) = 0, \quad 0 < t < \tau, \quad \epsilon > 0 \quad (1)$$

with initial conditions

$$y(0, \epsilon) = 0, \quad y'(0, \epsilon) = \frac{\bar{c}}{\epsilon}, \quad \bar{c} \in \mathcal{R} \setminus \{0\} \quad (2)$$

where

- $a [0, \tau] \rightarrow (0, \infty)$  is  $C^1$  nonincreasing function,
- $p$  is  $C^1$  decreasing function,  $p(t) \geq 0$  on  $[0, \tau]$ ,
- $f$  is a continuous odd function with exactly three simple zeros

$$f(-y_0) = f(0) = f(y_0) = 0, \quad f'(0) < 0 \quad \text{and} \quad \lim_{y \rightarrow \infty} \frac{f(y)}{y} < \infty.$$

Without loss of generality, we may consider that  $f$  is increasing on  $(y_0, \infty)$  and as a illustrative example, we can take  $f(y) = y - 2 \arctan y$  (i.e.  $(\pm y_0, 0)$  be the nonhyperbolic equilibria). For hyperbolic theory see [3] (in this work the author's consider the FitzHugh-Nagumo equation with a specific choice of  $f$ ,  $f(y) = y(1-y)(y-a)$ , where  $a > 0$  is a real parameter), [2], for example.

For the singular problem, that is, when  $\epsilon \rightarrow 0^+$ , the considerations as below may be relevant to the study of equilibrium solutions of the scalar parabolic reaction

- diffusion equation with decreasing diffusion. Also, we remark that for  $p \geq 0$  the corresponding equation is not dissipative. Moreover, this class of equations has special significance in connection with applications involving nonlinear vibrations and chaos (e.g. references [4], [5], [7]).

The problems of this form have been previously considered by Rocha [6] and Angenent *et al.* [1]. In [6],  $p(t) \equiv -1$ . However, the situation is more complicated when  $p(t) \geq 0$  from the existence of the zeros of solutions point of view. It follows from the fact that the linearized version of (1) do not admit an oscillatory solution for  $p(t) \geq 0$ , unlike the case  $p(t) < 0$ .

The goal of this paper is to prove the existence of decreasing sequence  $\{\epsilon_n\}_{n=n_0}^\infty$ ,  $\epsilon_n \rightarrow 0^+$  such that the corresponding solutions of (1), (2) have  $n$  zeros on  $(0, \tau)$  for every  $n \geq n_0$ ,  $n \in \mathcal{N}$  and  $y(\tau, \epsilon_n) = 0$ .

*Notation.*

$$\bar{f}(y) = \frac{f(y)}{y} \text{ for } y \neq 0 \text{ and } \bar{f}(0) = f'(0);$$

$$V(y) = \int_0^y f(u) du;$$

$z_{[0, \tau]}(y)$  denote the zero number of the nontrivial solution  $y$  of (1) on  $(0, \tau)$ ;

$-Y_{s,\epsilon}, Y_{s,\epsilon}$  ( $-Y_{s,\epsilon} < Y_{s,\epsilon}$ ) denote the real roots of equation

$$p(s)V(y) = \hat{H}(s, y(s, \epsilon), w(s, \epsilon), \epsilon) - r(s, \epsilon)$$

for  $s \in [0, \tau]$  (for definition of  $\hat{H}$  and  $r$  see below);

$-Y_0, 0, Y_0$  ( $-Y_0 < 0 < Y_0$ ) denote the real roots of equation  $V(y) = 0$ ;

$d_I(y)$  denotes spacing between two successive zero numbers of  $y$  on the interval  $I$ .

The rest of this paper is organized as follows. In Section 2 is explained technique necessary for the understanding of the paper and in Section 3 is formulated the theorem which is the main result.

## 2 Preliminaries

In order to apply the standard approach for the study of the solutions of (1) we introduce the variable  $w = \epsilon ay'$  and write this equation in the system form

$$y' = \frac{1}{\epsilon a} w, \quad w' = -\frac{1}{\epsilon a} p(t)f(y) - \frac{a'}{a} w. \quad (3)$$

If we consider the function

$$\hat{H}(t, y, w, \epsilon) = H(t, y, w) + r(t, \epsilon)$$

where

$$H(t, y, w) = \frac{1}{2} w^2 + p(t)V(y), \quad r(t, \epsilon) = - \int_0^t \left\{ p'(s) \int_0^{y(s, \epsilon)} f(u) du \right\} ds$$

and compute its derivative along the solutions of (3), we have

$$\begin{aligned} \dot{\hat{H}}(t, y, w, \epsilon) &= ww' + p(t)f(y)y' + p'(t) \int_0^y f(u) du + r'(t, \epsilon) \\ &= w \left[ -\frac{1}{\epsilon a} p(t)f(y) - \frac{a'}{a} w \right] + p(t)f(y)y' + p'(t) \int_0^y f(u) du + r'(t, \epsilon) = -\frac{a'}{a} w^2. \end{aligned}$$

Hence, we conclude that  $\hat{H}$  is monotone, nondecreasing function and  $\hat{H}|_{t=0} = \frac{1}{2}(a(0)\bar{c})^2 > 0$  implies that  $\hat{H} > 0$  on  $[0, \tau]$  for every  $\epsilon$ . We use the level curves of  $\hat{H}$  to characterize the trajectories of (3). One first draws the  $(t, y) - p(t)V(y)$  profile and then graphically draws the trajectory  $(y, w)$  with

$$w = \pm \left( 2 \left( \hat{H} - r(t, \epsilon) - p(t)V(y) \right) \right)^{\frac{1}{2}}$$

extending it as long as  $w$  remains real (i.e., until  $p(t)V(y)$  exceeds  $\hat{H} - r$ ). It is instructive for the future to keep in the mind the phase-portraits of the system (3) with  $\hat{H} - r \geq 0$  (see Figure 1).

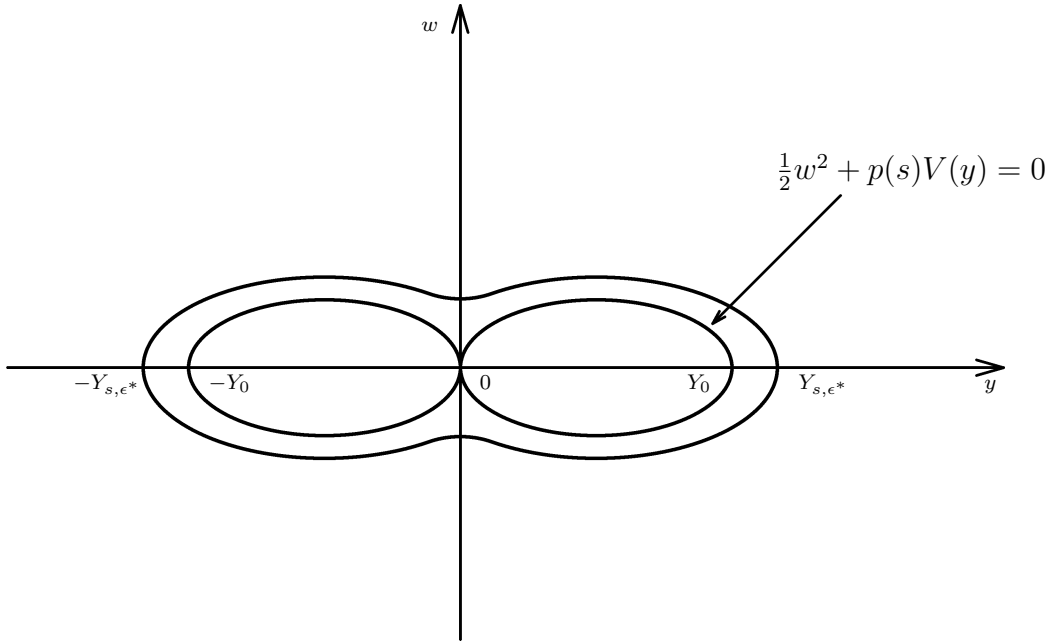


Figure 1: Intersection a 3-dimensional manifold  $\frac{1}{2}w^2 + p(s)V(y) = \hat{H} - r$  with the subspace  $(t, \epsilon) = (s, \epsilon^*)$  of  $(t, y, w, \epsilon)$ -space for  $\hat{H} - r > 0$ .

Let us introduce the new variable  $v = \epsilon a^2 y'$  and write (1) in the system form

$$y' = \frac{1}{\epsilon a^2} v, \quad v' = -\frac{1}{\epsilon} p(t) f(y)$$

Then, expressing  $(y, v)$  in polar coordinates,  $y = r \cos \gamma$ ,  $v = -r \sin \gamma$  (obviously,  $y^2 + v^2 > 0$  for every nontrivial solution of (1)) we obtain the following

differential equation for  $\gamma$ :

$$\gamma' = \frac{1}{\epsilon} \left[ \frac{1}{a^2(t)} \sin^2 \gamma + p(t) \bar{f}(y(t, \epsilon)) \cos^2 \gamma \right] \quad (4)$$

$\gamma(0) = \frac{\pi}{2}$  (for  $\bar{c} < 0$ ) or  $\gamma(0) = \frac{3\pi}{2}$  (for  $\bar{c} > 0$ ). It is clear, that  $\gamma(1, \epsilon) > (2k+1)\frac{\pi}{2}$ ,  $k \in \mathcal{N}$  implies that  $z_{[0, \tau]}(y(t, \epsilon)) > k$  (for  $\bar{c} < 0$ ).

### 3 Main result

We precede the main result of this paper by important Lemma.

*Lemma.*  $\hat{H} - r \geq 0$  on  $[0, \tau]$ ,  $\hat{H} - r > 0$  for  $y \in (-Y_0, Y_0)$ .

*Proof.* The statement follows immediately from the inequalities

$$\left( \hat{H} - r \right)' = p'V - \frac{a'}{a} w^2 \geq p'V > 0 \text{ for } y \in (-Y_0, Y_0)$$

and

$$\hat{H} - r > 0 \text{ for } y \in [-Y_{t, \epsilon}, Y_{t, \epsilon}] \setminus [-Y_0, Y_0].$$

Now we formulate the main result of this paper.

*Theorem.* Consider the problem (1),(2). Assume that

$$\frac{f(y)}{y} > \frac{2V(y)}{y^2} \text{ for every } y \in [-y_0, y_0] \setminus \{0\}. \quad (5)$$

Then there is a decreasing sequence  $\{\epsilon_n\}_{n=n_0}^\infty$ ,  $\lim_{n \rightarrow \infty} \epsilon_n = 0$  such that

- (i) The corresponding solution of (1), (2) has  $z_{[0, \tau]}(y(t, \epsilon_n)) = n$ .
- (ii)  $h_1(t, \epsilon_n) \leq d_I(y(t, \epsilon_n)) \leq h_2(t, \epsilon_n)$ , where  $h_1(t, \epsilon_n) = O(\epsilon_n)$ ,  $h_2(t, \epsilon_n) = O(\epsilon_n)$  and  $I \subset [0, \tau]$  is connected and closed.

*Proof.* Using the identity  $\cos^2 \gamma + \sin^2 \gamma = 1$ , from (4) we obtain

$$\gamma' = \frac{1}{\epsilon} \left[ \frac{1}{a^2} + \cos^2 \gamma \left( p(t) \bar{f}(y) - \frac{1}{a^2} \right) \right].$$

Let  $\xi \in [0, \tau]$ . The definition of polar coordinates implies that  $\cos^2 \gamma = \frac{y^2}{y^2+v^2} = \frac{y^2}{y^2+a^2w^2}$ . Thus,

$$\left| \cos^2 \gamma \left( p\bar{f} - \frac{1}{a^2} \right) \right| = \left| \frac{y^2 (p\bar{f} - \frac{1}{a^2})}{y^2 + v^2} \right| = \left| \frac{y^2 (p\bar{f} - \frac{1}{a^2})}{y^2 + 2a^2(\hat{H} - r - pV)} \right|$$

for  $y \in [-Y_{\xi, \epsilon}, Y_{\xi, \epsilon}]$ .

Because  $\hat{H} - r > 0$ , there is a positive constant  $\kappa_1$  independent of  $\epsilon$ ,  $\kappa_1 < y_0$  such that

$$\left| \cos^2 \gamma \left( p(\xi)\bar{f} - \frac{1}{a^2(\xi)} \right) \right| < \frac{1}{2a^2(\xi)}$$

for  $|y| < \kappa_1$ . Further for  $|y| \geq \kappa_1$ , taking into consideration the conclusions of Lemma, we obtain

$$\left| \cos^2 \gamma \left( p(\xi)\bar{f} - \frac{1}{a^2(\xi)} \right) \right| \leq \left| \frac{y^2 (a^{-2}(\xi) - p(\xi)\bar{f}(y))}{y^2 - 2a^2(\xi)p(\xi)V(y)} \right|.$$

From the condition (5) we conclude that

$$0 < \frac{a^{-2}(\xi) - p(\xi)\bar{f}(y)}{a^{-2}(\xi) - \frac{2p(\xi)V(y)}{y^2}} < 1$$

and after little arrangement we get

$$0 < \frac{y^2 (a^{-2}(\xi) - p(\xi)\bar{f}(y))}{y^2 - 2a^2(\xi)p(\xi)V(y)} < \frac{1}{a^2(\xi)}$$

for  $\kappa_1 \leq |y| \leq y_0 + \kappa_2 < Y_0$ , where  $\kappa_2$  is a sufficiently small positive constant.

Let  $\tilde{c}(\xi) = \min \{ \tilde{c}_1(\xi), \tilde{c}_2(\xi) \}$ , where

$$\tilde{c}_1(\xi) = \min \left\{ \frac{1}{a^2(\xi)} - \frac{y^2 (a^{-2}(\xi) - p(\xi)\bar{f}(y))}{y^2 - 2a^2(\xi)p(\xi)V(y)}, |y| \leq y_0 + \kappa_2 \right\},$$

$$\tilde{c}_2(\xi) = \min \left\{ \frac{1}{a^2(\xi)} \sin^2 \gamma + p(\xi)\bar{f}(y) \cos^2 \gamma, y_0 + \kappa_2 \leq |y| \leq Y_{\xi, \epsilon}, \gamma \in \mathcal{R} \right\}$$

and we define for  $t \in [0, \tau_0)$  the function

$$c(t) = \min \{ \tilde{c}(\xi); \xi \in [0, t] \}.$$



Clearly,  $c(t)$  is a positive nonincreasing function on  $[0, \tau]$  except that  $c(\tau) = 0$  if  $p(\tau) = 0$ . Hence,  $\gamma' \geq \frac{c(t)}{\epsilon}$  for  $t \in [0, \tau]$ . Since  $c(t)$  is independent of  $\epsilon$  we conclude that, by taking  $\epsilon$  sufficiently small,  $\gamma$  can be made arbitrarily large,  $\gamma(\tau, \epsilon) \geq \frac{\pi}{2} + \frac{1}{\epsilon} \int_0^\tau c(u) du$  for  $\bar{c} < 0$ .

Moreover, setting  $c^*(t) = a^{-2}(t) + \sup \{p(t)\bar{f}(y); t \in [0, \tau], y \in \mathcal{R}\}$  on  $[0, \tau]$ , we get  $\gamma' \leq \frac{c^*(t)}{\epsilon}$  for  $t \in [0, \tau]$ . The continuity of  $\gamma$  with respect to a parameter  $\epsilon$ ,  $\epsilon \neq 0$  we obtain such values  $\epsilon$  that  $\gamma(\tau, \epsilon_n) = \frac{\pi}{2} + \pi n$  for every  $n \geq n_0$ . The corresponding solution of (1), (2) has a zero number  $z_{[0, \tau]}(y(t, \epsilon_n)) = n - 1$ . This completes the proof of statement (i). Integrating between two successive zero numbers of solution  $y(t, \epsilon)$  on an interval  $I \subset [0, \tau]$  (if  $p(\tau) = 0$  then  $I \subset [0, \tau)$ ), we obtain an estimate of  $d_I(y(t, \epsilon_n))$  of the form

$$\epsilon_n \left( \frac{\pi}{c_I^*} \right) \leq d_I(y(t, \epsilon_n)) \leq \epsilon_n \left( \frac{\pi}{c_I} \right),$$

where  $c_I^* = \sup_{t \in I} c^*(t)$  and  $c_I = \inf_{t \in I} c(t)$ . Theorem is proven.

Remark. Oddness of  $f$  is considered in order to avoid technicalities, and is not essential. One modifies the theorem for a general function  $f$  with three undegenerate zeros and satisfying  $0 \leq \limsup_{y \rightarrow \infty} \frac{f(y)}{y} < \infty$ .

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# A note on Szegö's theorem

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## Abstract

In this note, it is shown that the conclusions of the classical Szegö's theorem on asymptotic distribution of eigen values of finite sections of multiplication operators, do not remain valid when the trigonometric basis is replaced by the Haar basis.

**Keywords :** Fourier, Operator, Toeplitz, Matrix, Haar.

**AMS Classification :** 47A58.

## 1. Introduction

The wellknown theorem of Szegö [7] on Toeplitz matrices states that if  $\lambda_1, \lambda_2, \dots, \lambda_N$  are the eigen values of the  $N \times N$  section of the matrix  $A = (a_{i-j})$ , where  $a_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx} dx$  is the  $k^{th}$  Fourier coefficient of the multiplier  $f$  in  $L^\infty(-\pi, \pi)$ , and  $F : R \rightarrow R$  any function Riemann integrable on  $[m, M]$  where  $m = \text{essential infim } f$ , and  $M = \text{essential sup } f$ , then

$$\lim_{N \rightarrow \infty} \frac{\sum_{k=1}^N F(\lambda_k)}{N} = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(f(x)) dx. \quad (1)$$

In this note it is shown that an, analogue of the theorem in which the trigonometric basis is replaced by the Haar basis  $\{h_{m,n}\}$ , does not hold when it is ordered suitably. More specifically the problem considered is the validity of asymptotic formula (1) for the case when  $A = (a_{i,j})$  is the matrix of the multiplication operator  $A$  on  $L^2[0, 1]$ ,

$$A(g) = fg$$

with respect to the lexicographically ordered Haar basis.

It is also observed that when the same operator taken here is considered with respect the Haar system under a different ordering, the limit in formula (1) can exist.

## 2. Lexicographic ordering of Haar basis

The lexicographically ordered Haar basis can be represented as a sequence  $\{\phi_0, \phi_1 \cdots\}$  of functions where

$$\phi_n(x) = h_{m,r}^{(x)} = \begin{cases} 2^{m/2}, & \frac{r}{2^m} \leq x < \frac{r+1/2}{2^m} \\ -2^{m/2}, & \frac{r+1/2}{2^m} \leq x < \frac{r+1}{2^m} \\ 0 & \text{otherwise} \end{cases}$$

where  $n = 2^m + r$ ,  $0 \leq r < 2^m$  ( $m \geq 0$ ) and  $\phi_0(x) \equiv 1$ .

The main result of this note is the following proposition.

## 3. Proposition

Let  $T_{\phi_1}$  be the multiplication operator on  $L_2[0, 1]$  with  $\phi_1$  as multiplier. Then the asymptotic formula (1) is not satisfied when the Trigonometric basis is replaced by the lexicographically ordered Haar basis.

*Proof.* Let  $A = (a_{ij})$ , where  $a_{ij} = \int_0^1 \phi_1(x)\phi_l(x)\phi_j(x)dx$ . Then

$$a_{ij} = \begin{cases} +1, & i = j = n, \ 0 \leq r < 2^{m-1}, \\ & n = 2^m + r, \\ -1, & 2^{m-1} \leq r < 2^m \\ 0, & i \neq j, \ i = j = 1 \end{cases}$$

□

Now consider the following sequences  $A_N$  of sections of  $A$  when

- (1)  $N = 2^m - 1$ ,  $m = 0, 1, 2, 3 \dots$ , and
- (2)  $N = 2^{m-1} + 2^m - 1$ ,  $m = 0, 1, 2, 3 \dots$

In (1) the eigen values  $+1$  and  $-1$  each have multiplicity  $\frac{N-1}{2}$ . On the other hand  $+1$  has multiplicity  $\frac{2N-1}{3}$  and  $-1$  has multiplicity  $\frac{N-2}{3}$  for sections of the type (2).

Now let  $F : R \rightarrow R$  be a fraction which is Riemann integrable on  $[0, 1]$ , be each that  $F(1) \neq F(-1)$ . Then

$$\lim_{N \rightarrow \infty} \sum_{r=1}^N \frac{F(\lambda_k)}{N} = \begin{cases} \frac{F(1)+F(-1)}{2}, & \text{for the type (1) and} \\ \frac{2F(1)+F(-1)}{3}, & \text{for the type (2)} \end{cases}$$

#### 4. Remarks

It is a matter of curiosity to know the outcome when the multiplier is  $\phi_k$  for  $k > 1$ . Let  $\phi_k = h_{m,n}$ . It is not surprising to see that the conclusions are the same. This is illustrated in the table 1 which gives multiplicities of the corresponding eigen values for various sequences of sections. The details of computation are omitted. In table 1 the multiplicities of various sequences of sections are given by positive integers of the form  $N = 2^r + p$ ,  $\frac{p+1}{2^r} \leq 1$ . Observe that the eigen values of this multiplication operator are  $2^{\frac{m}{2}}$ ,  $-2^{\frac{m}{2}}$  and 0.

TABLE 1

Value of $p$	Mul. of e.v. $2^{m/2}$	Mul. Of e.v.- $2^{m/2}$	Mul.Of e.v. 0	$\text{Lt}_{t \rightarrow \infty} \frac{u(\lambda_1)+u(\lambda_2)+\dots+u(\lambda_N)}{N}$
0	$2^{r-m-1}-1$	$2^{r-m-1}-1$	$(2^{r-m}-1)(2^m-1)+1$	$\frac{u(2^{m/2})+u(-2^{m/2})}{2^{m+1}} + \frac{2^m-1}{2^m}u(0)$
$n$	$2^{r-m-1}-1$	$2^{r-m-1}-1$	$(2^{r-m}-1)(2^m-1)+n+1$	$\frac{u(2^{m/2})+u(-2^{m/2})}{2^{m+1}} + \frac{2^m-1}{2^m}u(0)$
$(n-5)$ $2^{r-m}$	$2^{r-m-1}-1$	$2^{r-m-1}-1$	$(2^{r-m}-1)(2^m-1)+(n-5)2^{r-m}+1$	$\frac{u(2^{m/2})+u(-2^{m/2})}{2^{m+1}+(n-5)2} + \frac{2^m+n-6}{2^m+n-5}u(0)$
$n2^{r-m}-1$	$2^{r-m-1}-1$	$2^{r-m-1}-1$	$(2^{r-m}-1)(2^m-1)+n2^{r-m}$	$\frac{u(2^{m/2})+u(-2^{m/2})}{2^{m+1}+2n} + \frac{2^m+n-1}{2^m+n}u(0)$
$n2^{r-m}+2^{r-m-3}$	$2^{r-m-1}+2^{r-m-3}$	$2^{r-m-1}-1$	$(2^{r-m}-1)(2^m-1)+n2^{r-m}$	$\frac{5u(2^{m/2})+4u(-2^{m/2})+(2^m-1)8+8nu(0)}{2^{m+3}+8n+1}$
$(n+1)$ $2^{r-m}$	$2^{r-m}-1$	$2^{r-m}-1$	$(2^{r-m}-1)(2^m-2)+(n+1)2^{r-m}$	$\frac{u(2^{m/2})+u(-2^{m/2})+2^{m+1}+n-1u(0)}{2^{m+1}+n+1}$
$(2n+1)$ $2^{r-m-1}$	$2^{r-m}-1$	$2^{r-m-1}+n2^{r-m}$	$(2^{r-m}-1)(2^m-1)+n2^{r-m}$	$\frac{2u(2^{m/2})+u(-2^{m/2})+2^{m+1}+2n-2u(0)}{2^{m+1}+2n+1}$
$2^r-1$	$2^{r-m}-1$	$2^{r-m}-1$	$(2^{r-m}-1)(2^m-2)+2^r-1$	$\frac{u(2^{m/2})+u(-2^{m/2})}{2^{m+1}} + \frac{2^m-1}{2^m}u(0)$

We conclude this note by showing that there is an ordering for the Haar system such that the averages in the asymptotic formula (1) of various sections of  $T_{\phi_1}$ , converges. The following proposition gives this

## 5. Proposition

Let  $H$  be the Haar system in  $L^2[0, 1]$  ordered as

$$\{h_{00}, h_{10}, \dots, h_{n0}, h_{n-1,1}, h_{n-2,2}, \dots, h_{n-k,k}, \dots\}, \frac{k+1}{2^{n-k}} \leq 1.$$

Let  $B$  be the matrix of the multiplication operator  $T_{\phi_1}$ , in  $L^2[0, 1]$  with respect to this basis. If  $\lambda_1, \lambda_2, \dots, \lambda_N$  are the eigen values of the  $N^{th}$  section  $B_N$  of  $B$  and  $F$  a real function in  $R$  which is Riemann integrabilities on  $[0, 1]$ , then

$$\lim_{N \rightarrow \infty} \frac{\sum_{k=1}^N F[\lambda_k]}{N} = F(1)$$

*Proof.* Let  $N$  be a positive integer and let  $h_{m-k,k}$  be the  $N$ th basis element. We show that  $N = O(m^2)$ . For positive integers  $n$  and  $k'$ ,  $0 \leq n \leq m$ ,

$$\begin{aligned} \frac{k'+1}{2^{n-k}} &\leq 1 \\ \Leftrightarrow \log(k'+1) + k' &\leq n \\ \Rightarrow k' &\geq \frac{n-1}{2} \\ \therefore N &\geq \sum_{n=0}^m \frac{n-1}{2} = O(m^2) \end{aligned}$$

Also the only eigen values of  $B_N$  are  $+1$  and  $-1$ . Let  $N_1$  and  $N_{-1}$  be the multiplicities of the eigen values  $+1$  and  $-1$  of  $B_N$ . We show that

$$\frac{N_{-1}}{N} \rightarrow 0 \text{ as } N \rightarrow \infty.$$

It is clear that  $N_{-1} = \text{cardinality } \cup_{m \leq n} \{k' \mid \text{support } h_{m-k',k'} \subseteq [\frac{1}{2}, 1], k' \leq m\}$ .

Suppose

$$\begin{aligned} h_{m-k', k'} \subseteq [\frac{1}{2}, 1] &\Leftrightarrow \frac{k}{2^{n-k'}} \geq \frac{1}{2} \\ &\Leftrightarrow k' 2^{k'+1} \geq 2^m \Leftrightarrow k' + 1 + \log k' \geq m \\ \therefore k' &> m - \log m - 1 \end{aligned}$$

Hence the number of such  $k'$  is atmost  $\log m + 1$  for each  $m$ .

$$\therefore N_{-1} \leq \sum_{m=1}^n (\log m + 1) \leq n(\log n + 1)$$

This completes the proof. □

## 6. Remarks

The authors wish to point out the works of Kent. E. Morrison [6] where the so called Walsh-Toeplitz matrices and their eigen value distribution are analyzed.

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## Exponential decay of $2 \times 2$ operator matrix semigroups

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ABSTRACT. We show the exponential decay of certain  $2 \times 2$  operator matrix semigroups in Hilbert spaces. Applications are given to several damped wave equations.

### 0. INTRODUCTION

Investigating the exponential stability of matrix semigroups is a difficult but interesting question. In fact, there are well-known examples for damped wave equations where the spectral bound does not equal the growth bound of the semigroup, see e.g. [Le] and [Re]. Hence, for these equations it not suffices to calculate the spectrum of the associated generator and a more detailed analysis is necessary.

The aim of this note is to give a sufficient condition for the exponential stability of matrix semigroups with estimates on the decay rates. This is done in the first section. Similar investigations were made by S.-Z. Huang in [Hu].

In the second section we give sufficient conditions for the well-posedness of complete second order Cauchy problems and apply our results of Section 1 in order to study the asymptotic behavior of the solutions.

Finally, in the last section we consider some typical applications which illustrate our abstract results obtained in the first two sections.

### 1. ASYMPTOTIC BEHAVIOR OF SOLUTIONS

We start by setting our general framework.

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Let  $C : D(C) \subset H \rightarrow H$  be a densely defined linear operator on a Hilbert space  $H$ . Moreover, let  $B : D(B) \subset H \rightarrow H$  and  $D : D(D) \subset H \rightarrow H$  be dissipative linear operators such that the operator

$$\mathcal{A}_0 := \begin{pmatrix} D & C \\ -C^* & B \end{pmatrix} \quad \text{with domain} \quad D(\mathcal{A}_0) := (D(C^*) \cap D(D)) \times (D(C) \cap D(B))$$

is densely defined on the Hilbert space  $\mathcal{W} := H \times H$  equipped with its canonical inner product. For wave equations we typically have  $D = 0$ , however to cover a wider range of applications we consider here the more general case.

One easily verifies that under these conditions  $\mathcal{A}_0$  is dissipative, hence closable, see [EN, Prop.II.3.14.(iv)], and we denote by  $\mathcal{A} := \overline{\mathcal{A}_0}$  its closure. Here in the terminology of dissipative, accretive and sectorial operators we follow [Ka].

Our main assumptions on the operators  $B$ ,  $C$  and  $D$  are the following.

**Hypothesis (H).** *There exist constants  $\gamma \geq 0$  and  $\delta > 0$  such that for all  $\begin{pmatrix} x \\ y \end{pmatrix} \in D(\mathcal{A}_0)$  we have*

$$(*) \quad \delta \min \{ \|x\|^2, \|y\|^2 \} \leq \operatorname{Re} \langle -Dx, x \rangle + \operatorname{Re} \langle -By, y \rangle$$

and

$$(**) \quad |\operatorname{Im} \langle Dx, x \rangle| + |\operatorname{Im} \langle By, y \rangle| \leq \gamma (\operatorname{Re} \langle -Dx, x \rangle + \operatorname{Re} \langle -By, y \rangle).$$

The stability estimates obtained later will depend on  $\delta$  and  $\gamma$ . Actually, it turns out that by choosing  $\delta$  greater and  $\gamma$  smaller our estimates will become sharper. Moreover, we note that for  $D = 0$  and  $B$  symmetric, Hypothesis (H) simplifies to the condition that  $B$  is negative definite.

To prove our main stability theorem, we need the following technical result.

**1.1 Lemma.** *Let  $0 < \varepsilon < \frac{\delta}{2}$  and  $\alpha \in (-\frac{\delta}{2} + \varepsilon, 0]$ . If*

$$\inf_{x \in D(\mathcal{A}), \|x\|=1} \|(\alpha + i\beta - \mathcal{A})x\| < \varepsilon,$$

then

$$|\beta| < \frac{(\varepsilon - \alpha)\gamma + 3\varepsilon}{\delta + 2(\alpha - \varepsilon)} \cdot \delta.$$

*Proof.* Let  $x := \begin{pmatrix} x \\ y \end{pmatrix} \in D(\mathcal{A})$  satisfy  $\|x\| = 1$  and  $\|(\alpha + i\beta - \mathcal{A})x\| < \varepsilon$ . Since  $D(\mathcal{A}_0)$  is a core for  $\mathcal{A}$  we may assume that  $x \in D(\mathcal{A}_0)$  and therefore obtain

$$(1.1) \quad \|(\alpha + i\beta - \mathcal{A})x\| = \left\| \begin{pmatrix} \alpha x + i\beta x - Dx - Cy \\ C^*x + \alpha y + i\beta y - By \end{pmatrix} \right\| < \varepsilon.$$

Thus  $|\langle (\alpha + i\beta - \mathcal{A})x, x \rangle| < \varepsilon$ , i.e.,

$$(1.2) \quad |\alpha + i\beta - \langle Dx, x \rangle + 2i \operatorname{Im} \langle x, Cy \rangle - \langle By, y \rangle| < \varepsilon$$

and hence

$$(1.3) \quad |\operatorname{Re} \langle Dx, x \rangle + \operatorname{Re} \langle By, y \rangle - \alpha| < \varepsilon.$$

This estimate combined with (\*) in Hypothesis (H) implies

$$(1.4) \quad \delta \min \{ \|x\|^2, \|y\|^2 \} \leq -\operatorname{Re}\langle Dx, x \rangle - \operatorname{Re}\langle By, y \rangle < \varepsilon - \alpha.$$

At this point we have to divide the proof in two cases.

*Case I:*  $\|y\|^2 < \frac{\varepsilon - \alpha}{\delta}$ . Then

$$\|x\|^2 = 1 - \|y\|^2 > 1 - \frac{\varepsilon - \alpha}{\delta},$$

hence the assumption on  $\alpha$  implies

$$1 - 2\|x\|^2 < 2\frac{\varepsilon - \alpha}{\delta} - 1 < 0,$$

and therefore

$$(1.5a) \quad |1 - 2\|x\|^2| > 1 - 2\frac{\varepsilon - \alpha}{\delta}.$$

On the other hand, we obtain from (1.1) the estimate

$$(1.6a) \quad \| -Dx - Cy + (\alpha + i\beta)x \| < \varepsilon$$

which gives by taking first the inner product with  $x$

$$| \langle -Dx - Cy + (\alpha + i\beta)x, x \rangle | < \varepsilon,$$

and then imaginary parts

$$(1.7a) \quad | -\operatorname{Im}\langle Dx, x \rangle + \operatorname{Im}\langle x, Cy \rangle + \beta\|x\|^2 | < \varepsilon.$$

Next, we combine (1.7a) with (1.2) and conclude

$$(1.8a) \quad \begin{aligned} 3\varepsilon &> |\beta(1 - 2\|x\|^2) + \operatorname{Im}\langle Dx, x \rangle - \operatorname{Im}\langle By, y \rangle| \\ &\geq |\beta| \cdot |1 - 2\|x\|^2| - |\operatorname{Im}\langle Dx, x \rangle - \operatorname{Im}\langle By, y \rangle|. \end{aligned}$$

Together with (1.5a), (\*\*) and (1.4) this implies

$$(1.9) \quad |\beta| \left( 1 - 2\frac{\varepsilon - \alpha}{\delta} \right) \leq 3\varepsilon + \gamma(-\operatorname{Re}\langle Dx, x \rangle - \operatorname{Re}\langle By, y \rangle) < 3\varepsilon + \gamma(\varepsilon - \alpha)$$

and the assertion follows.

*Case II:*  $\|x\|^2 < \frac{\varepsilon - \alpha}{\delta}$ . Then

$$\|y\|^2 = 1 - \|x\|^2 > 1 - \frac{\varepsilon - \alpha}{\delta},$$

and the proof follows analogously. Hence we collect only the corresponding formulas

$$(1.5b) \quad |1 - 2\|y\|^2| > 1 - 2\frac{\varepsilon - \alpha}{\delta},$$

$$(1.6b) \quad \|C^*x - By + (\alpha + i\beta)y\| < \varepsilon,$$

$$(1.7b) \quad |-\operatorname{Im}\langle Bx, x \rangle + \operatorname{Im}\langle x, Cy \rangle + \beta\|y\|^2| < \varepsilon$$

and

$$(1.8b) \quad \begin{aligned} 3\varepsilon &> |\beta(1 - 2\|y\|^2) - \operatorname{Im}\langle Dx, x \rangle + \operatorname{Im}\langle By, y \rangle| \\ &\geq |\beta| \cdot |1 - 2\|y\|^2| - |\operatorname{Im}\langle Dx, x \rangle - \operatorname{Im}\langle By, y \rangle|, \end{aligned}$$

which again imply (1.9). Since by (1.4) Cases I & II cover all possibilities the proof is complete.  $\square$

*Remark.* If  $D = 0$  the proof simplifies to Case I.

Next we observe that if  $\mathcal{A}$  is invertible, it is maximal dissipative and hence by the Lumer–Phillips theorem generates a contraction semigroup on  $\mathcal{W}$ . Our main result gives an estimate on the growth bound (or type)  $\omega_0(\mathcal{A})$  of this semigroup, cf. [EN, Def.I.5.6].

**1.2 Theorem.** *If Hypothesis (H) is satisfied and  $\mathcal{A}$  is invertible, then*

$$\omega_0(\mathcal{A}) \leq \max\left\{s(\mathcal{A}), -\frac{\delta}{2}\right\} < 0.$$

*In particular, the semigroup generated by  $\mathcal{A}$  is uniformly exponentially stable.*

*Proof.* Due to the Gearhart–Greiner–Prüß theorem, see [EN, Thm.V.1.11], it suffices to show that

$$(1.10) \quad \alpha + i\mathbb{R} \subset \rho(\mathcal{A}) \quad \text{and} \quad \sup_{\beta \in \mathbb{R}} \|R(\alpha + i\beta, \mathcal{A})\| < +\infty$$

for all  $\alpha > w$ , where  $w = \max\{s(\mathcal{A}), -\frac{\delta}{2}\}$ .

Since  $\mathcal{A}$  generates a contraction semigroup, we know that (1.10) is satisfied for  $\alpha > 0$ . Next recall that the boundary  $\partial\sigma(\mathcal{A})$  is always contained in the approximate point spectrum  $A\sigma(\mathcal{A})$ , see e.g. [EN, Prop.IV.1.10]. However, from Lemma 1.1 we conclude that for  $\alpha \in (-\frac{\delta}{2}, 0]$  and  $|\beta| > -\frac{\alpha\gamma\delta}{\delta+2\alpha}$  we have  $\alpha + i\beta \notin A\sigma(\mathcal{A})$  and therefore  $\alpha + i\beta \in \rho(\mathcal{A})$ . In particular, the invertibility of  $\mathcal{A}$  implies that  $s(\mathcal{A}) < 0$ . Next, for fixed  $\alpha \in (w, 0]$  we obtain again by Lemma 1.1 that the resolvent  $R(\alpha + i\beta, \mathcal{A})$  stays uniformly bounded in norm for  $|\beta| > 1 - \frac{\alpha\gamma\delta}{\delta+2\alpha}$ . Since it remains also uniformly bounded in norm on the compact set  $\{\alpha + i\beta : |\beta| \leq 1 - \frac{\alpha\gamma\delta}{\delta+2\alpha}\} \subset \rho(\mathcal{A})$  we obtain (1.10) which completes proof.  $\square$

If the spectral bound of  $\mathcal{A}$  is not known or difficult to calculate we have the following weaker estimate for  $\omega_0(\mathcal{A})$  in terms of the norm of the inverse of  $\mathcal{A}$ .

**1.3 Corollary.** *Assume that  $\mathcal{A}$  is invertible.*

(a) *If  $\gamma \neq 0$ , then*

$$\omega_0(\mathcal{A}) \leq w,$$

*where  $w \in (-\frac{\delta}{2}, 0)$  is the unique solution of the equation*

$$w^2 + \frac{w^2\gamma^2\delta^2}{(\delta + 2w)^2} = \|\mathcal{A}^{-1}\|^{-2}.$$

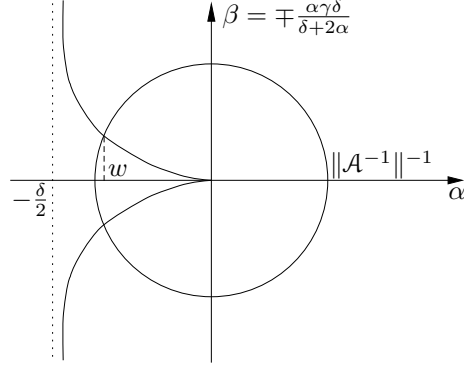


Figure 1

(b) If  $\gamma = 0$ , then

$$\omega_0(\mathcal{A}) \leq w := \max \left\{ -\frac{\delta}{2}, -\|\mathcal{A}^{-1}\|^{-1} \right\}.$$

*Proof.* The assertion follows from Lemma 1.1 as in the proof of Theorem 1.2 combined with the fact that  $\{\lambda \in \mathbb{C} : |\lambda| < \|\mathcal{A}^{-1}\|^{-1}\} \subset \rho(\mathcal{A})$ , cf. Figure 1.  $\square$

Finally, by a simple density argument we arrive at the following result.

**1.4 Corollary.** *Let  $\mathcal{A}$  be maximal dissipative and assume that  $\gamma \neq 0$ . If there exists  $\kappa > 0$  such that*

$$\|\mathcal{A}_0 x\| \geq \kappa \|x\| \quad \text{for all } x \in D(\mathcal{A}_0),$$

then

$$\omega_0(\mathcal{A}) \leq w,$$

where  $w \in (-\frac{\delta}{2}, 0)$  is the unique solution of the equation

$$(1.11) \quad w^2 + \frac{w^2 \gamma^2 \delta^2}{(\delta + 2w)^2} = \kappa^2.$$

We now turn our attention to complete second order Cauchy problems.

## 2. EXPONENTIAL ENERGY DECAY OF COMPLETE SECOND ORDER CAUCHY PROBLEMS

In this section we consider the complete second order abstract Cauchy problem

$$(ACP_2) \quad \begin{cases} u''(t) - Bu'(t) + Au(t) = 0, & t \geq 0, \\ u(0) = u_0, & u'(0) = u_1 \end{cases}$$

on a Hilbert space  $H$ , where we assume that  $A$  is given by  $A = C^*C$  for a densely defined and invertible operator  $C$  on  $H$ . This implies that  $A$  is self-adjoint and positive definite, see [We, Thm.5.39], and the standard reduction  $u := \begin{pmatrix} C u \\ u' \end{pmatrix}$  transforms  $(ACP_2)$  into the first order system

$$(ACP) \quad \begin{cases} u'(t) = \mathcal{A}u(t), & t \geq 0, \\ u(0) = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \end{cases}$$

for the operator matrix  $\mathcal{A} := \begin{pmatrix} 0 & C \\ -C^* & B \end{pmatrix}$ , cf. [XL, Chap.6.4].

In order to apply our results from Section 1, we further suppose that  $D(C) \cap D(B)$  is dense in  $H$  and that there exist constants  $\gamma \geq 0$  and  $\delta > 0$  such that

$$(2.1) \quad |\operatorname{Im}\langle By, y \rangle| \leq \gamma \operatorname{Re}\langle -By, y \rangle \quad \text{and} \quad \delta \|y\|^2 \leq \operatorname{Re}\langle -By, y \rangle$$

for all  $y \in D(B)$ . We note that the optimal choice for  $\delta$  in (2.1) is given by

$$\delta_0 := \inf \{ \operatorname{Re}\langle -By, y \rangle : y \in D(B), \|y\| = 1 \}.$$

Then Hypothesis (H) in Section 1 is satisfied and we define the operator  $\mathcal{A}$  (as in Section 1 for  $D = 0$ ) as the closure of the densely defined and dissipative matrix

$$\mathcal{A}_0 := \begin{pmatrix} 0 & C \\ -C^* & B \end{pmatrix} \quad \text{with} \quad D(\mathcal{A}_0) := D(C^*) \times (D(C) \cap D(B))$$

on the Hilbert space  $\mathcal{W} := H \times H$ .

If  $\mathcal{A}$  is invertible, we can represent its inverse in matrix form

$$\mathcal{A}^{-1} = \begin{pmatrix} U & V \\ W & S \end{pmatrix} \in \mathcal{L}(\mathcal{W})$$

and (formally) conclude from  $\mathcal{A}^{-1}\mathcal{A}_0 = \operatorname{Id}|_{D(\mathcal{A}_0)}$  that its entries satisfy

$$\begin{aligned} V &= -C^{*-1}, \quad S = 0, \quad W = C^{-1}, \quad \text{and} \\ Uz &= C^{*-1}BC^{-1}z \quad \text{for all } z \in D(BC^{-1}) = C(D(B) \cap D(C)). \end{aligned}$$

These operators give rise to a bounded inverse of  $\mathcal{A}$  if and only if  $C(D(B) \cap D(C))$  is dense in  $H$  and

$$(2.2) \quad Q := \overline{(C^*)^{-1}BC^{-1}} \in \mathcal{L}(H).$$

In particular, these conditions are satisfied if  $B$  is  $C$ -bounded. In this case solutions of (ACP) even give rise to strong solutions of (ACP<sub>2</sub>) in the sense of [Fa, Sec.VIII].

We are now in the position to state our first stability result.

**2.1 Corollary.** *Let  $C$  and  $B$  be densely defined linear operators on a Hilbert space  $H$  satisfying the conditions (2.1) and (2.2). If  $C$  is invertible, then  $\mathcal{A}$  generates a uniformly exponentially stable  $C_0$ -contraction semigroup on  $\mathcal{W}$  satisfying*

$$\omega_0(\mathcal{A}) \leq \max \left\{ s(\mathcal{A}), -\frac{\delta}{2} \right\}.$$

*Proof.* We define

$$(2.4) \quad \begin{aligned} \tilde{\mathcal{A}} &:= \begin{pmatrix} Id & 0 \\ 0 & C^* \end{pmatrix} \begin{pmatrix} 0 & Id \\ -Id & Q \end{pmatrix} \begin{pmatrix} Id & 0 \\ 0 & C \end{pmatrix}, \\ D(\tilde{\mathcal{A}}) &:= \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in H \times D(C) : x - QCy \in D(C^*) \right\}. \end{aligned}$$

Then  $\mathcal{A}_0 \subseteq \tilde{\mathcal{A}}$  and  $\tilde{\mathcal{A}}D(\mathcal{A}_0)$  is dense in  $\mathcal{W}$ . Moreover,  $\tilde{\mathcal{A}}$  is invertible and hence  $D(\mathcal{A}_0)$  is a core for  $\tilde{\mathcal{A}}$ , i.e.  $\mathcal{A} = \overline{\mathcal{A}_0} = \tilde{\mathcal{A}}$ . Since the assumptions on  $B$  and  $C$  imply that  $\mathcal{A}_0$  is dissipative we conclude by the Lumer–Phillips theorem that  $\mathcal{A}$  generates a contraction semigroup. By Theorem 1.2 this semigroup is uniformly exponentially stable and the desired estimate for  $\omega_0(\mathcal{A})$  holds.  $\square$

If it is impossible to determine the spectral bound  $s(\mathcal{A})$  we can use the following weaker result in order to estimate the growth bound  $\omega_0(\mathcal{A})$  of  $\mathcal{A}$ .

**2.2 Corollary.** *For*

$$(2.3) \quad \kappa^{-1} := \|Q\| + 2\|C^{-1}\|$$

*the following estimates for the growth bound  $\omega_0(\mathcal{A})$  of  $\mathcal{A}$  hold.*

(a) *If  $\gamma \neq 0$ , then*

$$\omega_0(\mathcal{A}) \leq w,$$

*where  $w \in (-\frac{\delta}{2}, 0)$  is the unique solution of equation (1.11).*

(b) *If  $\gamma = 0$ , then*

$$\omega_0(\mathcal{A}) \leq \max \left\{ -\kappa, -\frac{\delta}{2} \right\}.$$

*Proof.* The statements follow from Corollaries 1.3, 1.4 and 2.1, resp., and the estimate

$$\begin{aligned} \|\mathcal{A}^{-1} \begin{pmatrix} x \\ y \end{pmatrix}\| &= (\|Qx - C^{*-1}y\|^2 + \|C^{-1}x\|^2)^{\frac{1}{2}} \\ &\leq \|Qx - C^{*-1}y\| + \|C^{-1}x\| \\ &\leq \|Q\| \cdot \|x\|_C + \|C^{*-1}\| \|y\| + \|C^{-1}\| \cdot \|x\| \\ &\leq ((\|Q\| + \|C^{-1}\|)^2 + \|C^{*-1}\|^2)^{\frac{1}{2}} \cdot (\|x\|^2 + \|y\|^2)^{\frac{1}{2}} \\ &\leq (\|Q\| + 2\|C^{-1}\|) \cdot (\|x\|^2 + \|y\|^2)^{\frac{1}{2}} \end{aligned}$$

for all  $\begin{pmatrix} x \\ y \end{pmatrix} \in D(\mathcal{A}_0)$ . □

Finally we show that in many cases our result is optimal.

**2.3 Corollary.** *Under the conditions of Theorem 2.1 assume further that  $C$  and  $B$  are commuting normal operators. Then*

$$\omega_0(\mathcal{A}) = s(\mathcal{A}).$$

*Proof.* Using the spectral theorem, we can transform the operator  $\mathcal{A}$  to a matrix multiplier in an  $L^2$  space. Using the spectral characterization as in [Ho1] and the fact that  $s\left(\begin{smallmatrix} 0 & c \\ c^* & b \end{smallmatrix}\right) \geq \frac{b}{2}$  for every scalar matrix  $\begin{pmatrix} 0 & c \\ c^* & b \end{pmatrix}$ , it easily follows that for  $\delta = \delta_0$  one has  $s(\mathcal{A}) \geq -\frac{\delta_0}{2}$ . □

For a different treatment of complete second order equations with commuting normal operators we refer to [Sh].

### 3. APPLICATIONS

**3.1 Example.** (*Internally damped wave equation*, see [Go]) Let  $\Omega$  be a bounded connected domain in  $\mathbb{R}^n$  and let  $\Gamma$  be its boundary. We suppose  $\Gamma$  to be piecewise smooth and consisting of two closed parts  $\Gamma^0$  and  $\Gamma^1$  such that  $\Gamma = \Gamma^0 \cup \Gamma^1$  and  $\Gamma^0 \cap \Gamma^1 = \emptyset$ . Moreover, we denote by  $\nu$  the outer unit normal of  $\Gamma$ .

We shall be concerned with the internally damped wave equation

$$(IDW) \quad \begin{cases} y''(x, t) = \mu \Delta y'(x, t) + \Delta y(x, t), & (x, t) \in \Omega \times (0, \infty), \\ y(x, 0) = y_0(x), \quad y'(x, 0) = y_1(x), & x \in \Omega \end{cases}$$

with mixed boundary conditions

$$\begin{aligned} y(x, t) &= 0 \quad \text{on} \quad \Gamma^0 \times (0, \infty), \\ \frac{\partial y(x, t)}{\partial \nu} &= 0 \quad \text{on} \quad \Gamma^1 \times (0, \infty), \end{aligned}$$

where  $\mu > 0$  is a constant.

Then the energy of a solution  $y$  of (IDW) is given by

$$E(t) = \frac{1}{2} \int_{\Omega} \left( |y'(x, t)|^2 + |\nabla y(x, t)|^2 \right) dx.$$

In order to apply our abstract results we have to reformulate this problem as an abstract second order Cauchy problem. Here for the notations of Sobolev spaces we follow [Ad].

First we define the following operators on the Hilbert space  $H := L^2(\Omega)$ :

$$A := -\Delta, \quad B := \mu\Delta, \quad D(B) := D(A) := \left\{ y \in H^2 : y|_{\Gamma^0} = 0, \frac{\partial y}{\partial \nu}|_{\Gamma^1} = 0 \right\}.$$

As  $A$  and  $B$  are commuting normal operators, so are  $A^{\frac{1}{2}}$  and  $B$ , where we used that  $D(A^{\frac{1}{2}}) = H_{\Gamma^0}^1 := \{f \in H^1 : f|_{\Gamma^0} = 0\}$ , cf. [Tr].

Hence, we can calculate the exact decay rate by Corollary 2.3 as

$$\omega_0(\mathcal{A}) = s(\mathcal{A}) = \sup \left\{ \frac{-\mu\lambda + \operatorname{Re} \sqrt{\mu^2\lambda^2 - 4\lambda}}{2} : \lambda \in \sigma(A) \right\} = \max \left\{ -\frac{\mu\lambda_1}{2}, \frac{-1}{\mu} \right\},$$

where we denote by  $\lambda_1 > 0$  the first eigenvalue of  $A$ .

This is an optimal result and hence improves the suboptimal decay rate derived in [Go]. See [CT] for the analyticity of the semigroup associated to this example.

**3.2 Example.** (*Damped wave equation*, see [Tc], [Ko, Sec.1.3], [CZ1], [CZ2], [Lo])

Let  $\Omega$  be an open, bounded domain in  $\mathbb{R}^n$  with smooth boundary  $\partial\Omega$ . Consider the damped wave equation

$$(DWE) \quad \begin{cases} u_{tt}(t, x) + q(x)u_t(t, x) - \Delta u(t, x) = 0, & (t, x) \in \mathbb{R}_+ \times \Omega, \\ u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), & x \in \Omega, \\ u(t, x) = 0, & (t, x) \in \mathbb{R}_+ \times \partial\Omega \end{cases}$$

for initial values  $u_0, u_1 \in L^2(\Omega)$ . Here we assume that  $q : \Omega \rightarrow \mathbb{C}$  is measurable and satisfies

$$|\operatorname{Im} q(x)| \leq -\gamma \operatorname{Re} q(x) \quad \text{and} \quad \delta \leq -\operatorname{Re} q(x) \text{ a.e.}$$

for some constants  $\gamma \geq 0, \delta > 0$ .

In order to reformulate (DWE) as an abstract second order Cauchy problem we take  $H := L^2(\Omega)$ ,  $A := -\Delta$  with domain  $D(A) := H^2(\Omega) \cap H_0^1(\Omega)$  and  $B = M_q$  the multiplication operator with the function  $q$  and maximal domain, see [EN, Prop.I.4.10] or [We, 4.1.Exp.1].

Then  $\mathcal{A}_0$  satisfies (\*) and (\*\*) in Hypothesis (H). Hence in order to apply our results we have to look for conditions implying the boundedness of (the closure of) the operator  $A^{\frac{1}{2}}B$ .

To this end we use Sobolev imbedding theorems as in [Ad, Thm.5.4.C], that is,  $H_0^1(\Omega) \hookrightarrow C_b(\Omega)$  in the one dimensional case,  $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$  ( $p \geq 2$ ) in two dimensions and  $H_0^1(\Omega) \hookrightarrow L^q(\Omega)$  ( $q \geq 2n/(n-2)$ ) in higher dimensions. If we assume that  $q \in L^r(\Omega)$  with  $r \geq 2$  if  $n = 1$ ,  $r > 2$  if  $n = 2$  and  $r \geq n$  if  $n \geq 3$ ,



these imbeddings combined with Hölder's inequality imply that  $B$  is  $A^{\frac{1}{2}}$ -bounded. Hence, by Corollary 2.1 we obtain the exponential decay of the energy

$$E(t) = \frac{1}{2} \int_{\Omega} \left( |u'(t, x)|^2 + |\nabla u(t, x)|^2 \right) dx$$

of the solution  $u$  of (DWE).

To give a suboptimal estimate on the decay rate, we use formula (2.3) in Corollary 2.2 and obtain

$$(3.1) \quad \kappa^{-1} = \frac{\sqrt{-\lambda_1}}{2K\|q\|_{L^r} + 2},$$

where  $\lambda_1$  is the first eigenvalue of the Laplacian and  $K$  is the appropriate Sobolev imbedding constant.

In the one dimensional case, Cox and Zuaza derived the exact decay rates for bounded damping. For unbounded damping the only work known to us is [Tc] where other types of conditions ensure polynomial decay rates for not only strict dissipative damping.

**3.3 Example.** (*Reaction-diffusion system*, see [Na], [DD] and [dO]) Let  $\Omega$  be an open, bounded domain in  $\mathbb{R}^n$  with smooth boundary  $\partial\Omega$ . Consider the reaction-diffusion system

$$(RDS) \quad \begin{cases} u_t(t, x) = e(x)\Delta v(t, x) + q(x)v(t, x) + a(x)\Delta u(t, x) + b(x)u(t, x), \\ v_t(t, x) = -e^*(x)\Delta v(t, x) - q^*(x)u(t, x) + c(x)\Delta v(t, x) + d(x)v(t, x), \\ (t, x) \in \mathbb{R}_+ \times \Omega, \\ u(0, x) = u_0(x), \quad v(0, x) = v_0(x), \quad x \in \Omega, \\ u(t, x) = 0, \quad v(t, x) = 0 \quad (t, x) \in \mathbb{R}_+ \times \partial\Omega, \end{cases}$$

for initial values  $u_0, v_0 \in L^2(\Omega)$ .

Consider the following operators

$$D := a\Delta + M_b, \quad B := c\Delta + M_d, \quad C := e\Delta + M_q,$$

with maximal possible domain, where we again denote by  $M_f$  the multiplication operator with the function  $f$ .

Special cases of this equation are widely considered in the literature. It is proved in [DD] and [dO] that for  $a, c, e$  constant and  $b, d, q$  smooth the associated matrix operator generates an analytic semigroup.

We make assumptions on  $a, b, c, d, e$  and  $q$  implying that  $B, D$  and  $C$  generate analytic semigroups. We refer, e.g., to the dissertation of Holderrieth [Ho2, Example 1.6] for conditions implying that the operators  $a\Delta, c\Delta, e\Delta$  generate analytic semigroups. We assume that  $a, c, e \in L^\infty(\Omega)$ ,  $a^{-1}, c^{-1}, e^{-1} \in L^\infty(\Omega)$ , and that  $\operatorname{Re} a(x) > 0$ ,  $\operatorname{Re} c(x) > 0$  and  $\operatorname{Re} e(x) > 0$  a.e in  $\Omega$ . For the perturbation with the multiplicator, we refer here to two possibilities.

- (a) Assume that  $b, d, q \in L^r(\Omega)$  with  $r \geq 2$  if  $n = 1$ ,  $r > 2$  if  $n = 2$  and  $r \geq n$  if  $n \geq 3$ . This, combined with Hölder's inequality imply that the multiplication operator with these functions is  $\Delta$ -bounded with bound zero, see [Pa, Theorem 2.6.11]. Thus  $B, D$  and  $C$  generate analytic semigroups.

- (b) Let  $a, c, e$  be positive functions and assume that  $b, d, q \in L^r(\Omega)$  with  $r \geq \max\{p, \frac{n}{2}\}$  if  $n \neq 4$  and  $r > 2$  if  $n = 4$ . Assuming further that  $b, d, q \geq 0$ , we obtain by [AR, Example 2] that  $B, D$  and  $C$  generate (positive) analytic semigroups.

Assume further that either  $B$  or  $D$  is invertible and that the closure of the operator

$$\mathcal{A}_0 := \begin{pmatrix} D & C \\ -C^* & B \end{pmatrix}$$

has a bounded inverse. Then the solutions of (RDS) decay exponentially to zero whenever it is well-posed.

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# Sharp evaluation of the Prokhorov distance to zero under general moment conditions

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## Abstract

We determine the Prokhorov distance to the Dirac measure at zero for the families of distributions satisfying an arbitrary number of general moment conditions.

Key words: Prokhorov distance, radius, general moment condition, weak convergence, Chebyshev inequality.

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## 1 Introduction

The Prokhorov distance  $\pi$  of two probability measures defined on the Borel subsets of a Polish space with a metric  $\varrho$  is defined as

$$\pi(\mu, \nu) = \inf\{r > 0 : \mu(A) \leq \nu(A^r) + r, \text{ and } \nu(A) \leq \mu(A^r) + r \text{ for every closed } A\}, \quad (1)$$

where

$$A^r = \{x : \varrho(x, A) \leq r\}.$$

We clearly have  $\pi(\mu, \nu) \leq 1$  for arbitrary  $\mu$  and  $\nu$ . The Prokhorov distance generates the topology of weak convergence for probability measures. In

the case of standard Euclidean space, the weak convergence of a sequence of probability measures to a given one is equivalent to the convergence of respective distribution functions in all continuity points of the limiting distribution function. For the practical applications, it is important to evaluate the rate of weak convergence to degenerate measures. Without loss of generality, we often assume that the limiting distribution  $\delta_0$  is concentrated at 0. Since we usually easier control the rates of convergence of moments, it is of vital interest to know the relation between the Prokhorov distance of the degenerate limit and a random approximation, and distances of respective expectations of various functions. Therefore we consider the following problem.

For a positive integer  $k$ , consider a vector  $\varphi = (\varphi_1, \dots, \varphi_k) : \mathcal{R}^k \mapsto \mathcal{R}_+^k$  of gauge functions such that each  $\varphi_i$ ,  $1 \leq i \leq k$ , is nonnegative, continuous, vanishing at 0, nondecreasing on the positive half-axis, and symmetric about 0. Suppose that for a fixed  $\epsilon = (\epsilon_1, \dots, \epsilon_k) \in \mathcal{R}_+^k$ , we have

$$\int \varphi_i(x) \mu(dx) \leq \epsilon_i, \quad 1 \leq i \leq k. \quad (2)$$

Natural examples of gauge functions are  $\varphi_i(x) = |x|^{p_i}$ ,  $p_i > 0$ ,  $1 \leq i \leq k$ . The question is what is the greatest possible Prokhorov distance between  $\delta_0$  and  $\mu$ . In other words, we study the problem of determining the Prokhorov radius

$$D(\varphi, \epsilon) = \sup\{\pi(\mu, \delta_0) : \mu \in \mathcal{M}(\varphi, \epsilon)\}$$

for the set  $\mathcal{M}(\varphi, \epsilon)$  of probability measures satisfying (2) for fixed  $\varphi$  and  $\epsilon$ . The solution is given in Theorem 1. In Theorem 2, we show that imposing additional conditions

$$\left| \int \chi_j(x) \mu(dx) \right| \leq \delta_j, \quad 1 \leq j \leq l,$$

for some antisymmetric  $\mu$ -integrable functions (e.g.,  $\chi_j(x) = |x|^{q_j} \text{sgn}(x)$ ,  $q_j > 0$ ) does not affect the Prokhorov radius.

This significantly generalizes some earlier results, where at most three moment conditions with fixed functions  $\varphi$  and  $\chi$  were examined. Anastassiou [3] determined the Prokhorov radius for the family of distributions satisfying mean and variance conditions. Anastassiou and Rychlik [7] solved the problem under the additional condition on the third moment and the restriction to measures supported on the positive half-axis. Anastassiou and Rychlik [8] considered neighborhoods of zero defined by restrictions on the

first, second, and fourth moments. In a slightly different setup, Doukhan and Gamboa [9, 10] provided asymptotic evaluations of the Prokhorov radius of the set of measures on the unit interval described by some general moment conditions. Analogous Lévy radii for measures satisfying the first two moment conditions, and the conditions on the moments of two general functions forming with a constant a Chebyshev system were established in Anastassiou [1] and [2], respectively. Similar problem for the Kantorovich and Zolotarev metrics were studied by Anastassiou and Rachev [6] and Tardella [12], respectively. Some applications for the queueing models are indicated in Anastassiou and Rachev [5], and the most comprehensive review can be found in Anastassiou [4].

By now, precise calculations of Prokhorov radii were derived by means of the geometric moment method due to Kemperman [11]. The Prokhorov radius was first established for measures with fixed possible values of moments, and the outcomes were maximized over the set of all moment points satisfying inequality conditions. Usually, the maxima for the fixed moment points were expressed by means of sophisticated formulae in different subregions of the moment spaces. In consequence, the difficulty in application of the method significantly increased with increase of the number of moment conditions. In the case  $k = 3$ , the analysis of various functions of three variables was very complicated. The method proposed in the paper allows us to solve the maximization problem globally, and the number of moment conditions does not play any role. The arguments are simple, and the main tool is the well known Chebyshev inequality.

## 2 Results

**Theorem 1.** *Set  $\psi_i(x) = x\varphi_i(x)$ ,  $1 \leq i \leq k$ . Then*

$$D(\varphi, \epsilon) = \min\{1, \psi_i^{-1}(\epsilon_i), \quad 1 \leq i \leq k\}.$$

Note that each  $\psi_i$  is strictly increasing on  $\mathcal{R}_+$ , and its inverse is well defined there. In the special case

$$\varphi_i(x) = |x|^{p_i}, \quad 1 \leq i \leq k,$$

we have

$$D(\varphi, \epsilon) = \min\{1, \epsilon_i^{1/(p_i+1)}, \quad 1 \leq i \leq k\}.$$

PROOF. We first notice that the Prokhorov distance (1) of an arbitrary measure  $\mu$  to  $\delta_0$  has the form

$$\pi(\mu, \delta_0) = \inf\{r > 0 : \mu([-r, r]) \geq 1 - r\}. \quad (3)$$

If  $\mu$  has no atoms, then we can replace the inequality sign in (3) by the equality. For all  $1 \leq i \leq k$  and arbitrary positive  $r$ , the Chebyshev inequality implies

$$\int \varphi_i(x) \mu(dx) \geq \varphi_i(r) \mu(\mathcal{R} \setminus [-r, r]).$$

If  $\mu$  is continuous and  $r = \pi(\mu, \delta_0)$ , we have

$$r\varphi_i(r) = \psi_i(r) \leq \int \varphi_i(x) \mu(dx) \leq \epsilon_i,$$

and consequently

$$\pi(\mu, \delta_0) \leq \psi_i^{-1}(\epsilon_i), \quad 1 \leq i \leq k.$$

Since  $\pi(\mu, \delta_0)$  cannot exceed one by definition, we proved that

$$\begin{aligned} \tilde{D}(\varphi, \epsilon) &= \sup\{\pi(\mu, \delta_0) : \mu \in \mathcal{M}(\varphi, \epsilon), \text{ continuous}\} \\ &\leq \min\{1, \psi_i^{-1}(\epsilon_i), \quad 1 \leq i \leq k\}. \end{aligned} \quad (4)$$

We claim that

$$\tilde{D}(\varphi, \epsilon) = D(\varphi, \epsilon). \quad (5)$$

Here we use the fact that the set of continuous probability measures is  $\pi$ -dense which means that for every possible discontinuous  $\mu$  there exists a sequence of continuous  $\mu_n$ ,  $n \geq 1$ , such that the respective distribution functions tend to the distribution function of  $\mu$  at its all continuity points. For every  $1 \leq i \leq k$ , positive integer  $m$ , and the bounded continuous functions  $\varphi_{i,m} = \min\{\varphi_i, m\}$ , we have

$$\int \varphi_{i,m}(x) \mu_n(dx) \rightarrow \int \varphi_{i,m}(x) \mu(dx), \quad (6)$$

as  $n \rightarrow \infty$ . If all  $\mu_n \in \mathcal{M}(\varphi, \epsilon)$ , then

$$\int \varphi_{i,m}(x) \mu_n(dx) \nearrow \int \varphi_i(x) \mu_n(dx) \leq \epsilon_i, \quad (7)$$

as  $m \rightarrow \infty$ , which together with (6) imply that the same holds for  $\mu$ , and so  $\mu \in \mathcal{M}(\varphi, \epsilon)$ . Otherwise inequality in (7) is not satisfied for any  $\mu_n \rightarrow \mu$  and  $1 \leq i \leq k$ .



By (4) and (5), it suffices to show that

$$D(\varphi, \epsilon) \geq \min\{1, \psi_i^{-1}(\epsilon_i), \quad 1 \leq i \leq k\}. \quad (8)$$

Our proof is constructive. If the right-hand side is 1, then

$$\psi_i(1) = \varphi_i(1) \leq \epsilon_i, \quad 1 \leq i \leq k.$$

It suffices to take

$$\mu = \frac{1}{2}(\delta_1 + \delta_{-1}), \quad (9)$$

for which  $\pi(\mu, \delta_0) = 1$ , and

$$\int \varphi_i(x) \mu(dx) = \varphi_i(1) \leq \epsilon_i, \quad 1 \leq i \leq k.$$

This means that  $\mu \in \mathcal{M}(\varphi, \epsilon)$  and attains the bound.

Otherwise for some  $1 \leq j \leq k$ , the right-hand side of (8) amounts to  $\psi_j^{-1}(\epsilon_j) < 1$ , and we have  $\psi_j^{-1}(\epsilon_j) \leq \psi_i^{-1}(\epsilon_i)$  for  $i \neq j$ . Now we take

$$\mu = \frac{1}{2}\psi_j^{-1}(\epsilon_j) \left[ \delta_{\psi_j^{-1}(\epsilon_j)} + \delta_{-\psi_j^{-1}(\epsilon_j)} \right] + [1 - \psi_j^{-1}(\epsilon_j)]\delta_0. \quad (10)$$

We have

$$\begin{aligned} \int \varphi_j(x) \mu(dx) &= \frac{1}{2}\psi_j^{-1}(\epsilon_j) [\varphi_j(\psi_j^{-1}(\epsilon_j)) + \varphi_j(-\psi_j^{-1}(\epsilon_j))] \\ &= \psi_j^{-1}(\epsilon_j) \varphi_j(\psi_j^{-1}(\epsilon_j)) \\ &= \psi_j(\psi_j^{-1}(\epsilon_j)) = \epsilon_j, \end{aligned}$$

and likewise

$$\begin{aligned} \int \varphi_i(x) \mu(dx) &= \psi_j^{-1}(\epsilon_j) \varphi_i(\psi_j^{-1}(\epsilon_j)) \\ &= \psi_i(\psi_j^{-1}(\epsilon_j)) \leq \epsilon_i \end{aligned}$$

for  $i \neq j$ . It follows that  $\mu \in \mathcal{M}(\varphi, \epsilon)$ . Also, relation

$$\mu([-r, r]) = \begin{cases} 1 - \psi_j^{-1}(\epsilon_j), & \text{for } r < \psi_j^{-1}(\epsilon_j), \\ 1, & \text{for } r \geq \psi_j^{-1}(\epsilon_j), \end{cases}$$

together with (3) yield

$$\pi(\mu, \delta_0) = \psi_j^{-1}(\epsilon_j).$$

This ends the proof. ■

In fact, the above arguments do not change if we replace symmetric measures (9) and (10) by asymmetric ones with different mass distributions on the nonzero support points. Accordingly, the Prokhorov radius is attained by the measures concentrated on the nonnegative points only as well as the symmetric ones. However, we chose the symmetric representatives in order to prove a slightly refined version of Theorem 1.

Except for the conditions on moment of symmetric functions  $\varphi_i$ ,  $1 \leq i \leq k$ , we add similar ones for the moments of some antisymmetric functions  $\chi_j$ ,  $1 \leq j \leq l$ . Natural examples are even power functions. For  $\chi = (\chi_1, \dots, \chi_l)$  and  $\delta = (\delta_1, \dots, \delta_l) \in \mathcal{R}_+^l$ , we consider the family of probability measures

$$\mathcal{M}(\varphi, \chi, \epsilon, \delta) = \left\{ \mu : \begin{aligned} \int \varphi_i(x) \mu(dx) &\leq \epsilon_i, \quad 1 \leq i \leq k, \\ \left| \int \chi_j(x) \mu(dx) \right| &\leq \delta_j, \quad 1 \leq j \leq l \end{aligned} \right\}$$

and its Prokhorov radius

$$D(\varphi, \chi, \epsilon, \delta) = \sup \{ \pi(\mu, \delta_0) : \mu \in \mathcal{M}(\varphi, \chi, \epsilon, \delta) \}.$$

**Theorem 2.** *We have*

$$D(\varphi, \chi, \epsilon, \delta) = D(\varphi, \epsilon).$$

PROOF. Since  $\mathcal{M}(\varphi, \chi, \epsilon, \delta) \subset \mathcal{M}(\varphi, \epsilon)$ , we have

$$D(\varphi, \chi, \epsilon, \delta) \leq D(\varphi, \epsilon).$$

For symmetric measures (9) and (10),

$$\int \chi_j(x) \mu(dx) = 0, \quad 1 \leq j \leq l,$$

imply that  $\mu \in \mathcal{M}(\varphi, \chi, \epsilon, \delta)$ . Finally, relations

$$\pi(\mu, \delta_0) = D(\varphi, \epsilon) \geq D(\varphi, \chi, \epsilon, \delta)$$

complete the proof. ■

In the case of standard moments with

$$\begin{aligned}\varphi_i(x) &= x^{2i}, & 1 \leq i \leq k, \\ \chi_j(x) &= x^{2j-1}, & 1 \leq j \leq l,\end{aligned}$$

we obtain

$$D(\varphi, \chi, \epsilon, \delta) = \min\{1, \epsilon_i^{1/(2i+1)}, \quad 1 \leq i \leq k\}.$$

For  $(k, l) = (1, 1)$ ,  $(1, 2)$ , and  $(2, 1)$  in particular, we have the formulae derived in Anastassiou [3], and Anastassiou and Rychlik [7, 8], respectively.

Problems with moment constraints for functions  $\phi_i$  which are possibly neither symmetric nor antisymmetric need a more thorough analysis. However, under the restriction to the symmetric measures, they can be immediately solved by use of the results of Theorem 1. It merely suffices to replace original functions  $\phi_i(x)$  by the respective symmetrized versions  $\varphi_i(x) = \frac{1}{2}[\phi_i(x) + \phi_i(-x)]$ .

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# Asymptotic Stability of Linear Difference Equations of Advanced Type

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Necessary and sufficient conditions are obtained for the asymptotic stability of difference equations of advanced type of the form

$$x(n) - ax(n+1) + bx(n+k) = 0, n = 0, 1, ..$$

where  $a$  and  $b$  are arbitrary real numbers and  $k > 1$ .

For  $a = 1$ , we establish an analogue of a result by Levin and May.

KEY WORDS: Asymptotic stability; advanced type; difference equations.

## 1 Introduction

Delay idfference equations of the form

$$x(n+1) - ax(n) + bx(n-k) = 0, \quad n = 0, 1, \dots; (a \neq 0), \quad (1)$$

where  $k > 1$  is an integer and  $a, b \in \mathfrak{R}$ , have been extensively studied in the last decade, see for example Clark[1], Elaydi[2], Levin and May[6], Kuruklis[5], Kocic and Ladas [4]. When  $a=1$  and  $b$  is an arbitrary real number, and  $k$  is a positive integer, Levin and May showed that the zero solution of Eq(1) is asymptotically stable if and only if

$$0 < b < 2 \cos[k\pi/(2k+1)]. \quad (2)$$

Matsunaga and Hara [7] extended this result to the two dimensional system  $x(n+1) - x(n) + Bx(n-k) = 0$ , where  $B$  is a  $2 \times 2$  constant matrix. For the general case when  $a$  is any real number, Clark[1] gave an elegant proof to the following result: if  $|a| + |b| \leq 1$ , then the zero solution of Eq(1) is asymptotically stable. Later, Kuruklis [5] gave necessary and sufficient conditions for the zero solution of Eq(1) to be asymptotically stable. Moreover, his result includes as a special case the result of May and Levin cited above.

The main objective of this paper is to extend the above work to linear difference equations of advanced type of the form

$$x(n) - ax(n+1) + bx(n+k) = 0, \quad n = 0, 1, 2, \dots \quad (3)$$

where  $a$  and  $b$  are arbitrary reals and  $k > 1$  is an integer. These difference equations appeared in the book of Gyori and Ladas [3]. They may represent a mathematical model of species whose  $k$ th generation depends on the present and next generations. Moreover, difference equations of advanced type are usually associated with the study of differential equations with piecewise continuous argument such as

$$y' = Ay(t) + By([t+k]) \quad (4)$$

, where  $[ \ ]$  denotes the greatest integer function. If we let  $y_n(t)$  to be the solution of Eq(4) on the interval  $[n, n+1)$  and  $x(n) = y_n(n)$ , then Eq(4) may be transformed to Eq(3) (for more details see [8]). It is well known that the zero solution of Eq.(4) is asymptotically stable if and only if all the roots of its characteristic equation

$$b\lambda^k - a\lambda + 1 = 0 \quad (5)$$

are inside the unit disk. Equation (5) can be written equivalently as

$$c\mu^k - \mu + 1 = 0 \quad (6)$$

where  $c = b/a^k$  ( $a \neq 0$ ) and  $\mu = a\lambda$ . Hence all the roots of Eq.(5) are inside the unit disk if and only if all the roots of Eq.(6) are inside the disk  $|\mu| < |a|$ .

Our main result is Theorem 3.8 which provides necessary and sufficient conditions for the asymptotic stability of the zero solution of Eq.(3). As a consequence of this theorem we obtain an analogue of Levin and May's above celebrated result for advanced difference equations.

## 2 Preliminary Lemmas

**Lemma 2.1** . Let  $k > 1$  be an integer and  $a \neq 0$  be an arbitrary real number . Then the following inequality holds true

$$\frac{|a| - 1}{|a|^k} \leq \frac{(k-1)^{k-1}}{k^k} = \beta_k . \quad (7)$$

**Proof** . The equality sign holds for  $a = \pm \frac{k}{k-1}$  . Define the function  $f(a) = \frac{(k-1)^{k-1}}{k^k} a^k - a + 1$  . Hence  $f'(a) = \left(\frac{k-1}{k}\right)^{k-1} a^{k-1} - 1$  ,  $f''(a) = \frac{(k-1)^k}{k^{k-1}} a^{k-2}$  . Since  $f(\frac{k}{k-1}) = f'(\frac{k}{k-1}) = 0$  , it follows that  $a = \frac{k}{k-1}$  is a double root of  $f(a) = 0$  . Now if  $k$  is an even number , then  $f''(a) > 0$  ( for  $a > 0$  or  $a < 0$  ) . Therefore  $f(a) > 0$  and thus Inequality (7) holds. On the other hand if  $k$  is an odd number and  $a > 0$  , then  $f''(a) > 0$  and consequently,  $f(a) > 0$  and thus Inequality (7) holds . For  $a < 0$  and  $k$  is an odd number , we have  $f''(a) < 0$  . Since  $f'(-\frac{k}{k-1}) = 0$ , it follows that  $f(-\frac{k}{k-1}) = 2$  is the maximum value of  $f(a)$  . Hence  $f(a) \leq 2$ . This implies

$$\frac{(k-1)^{k-1}}{k^k} a^k \leq a + 1$$

and

$$\frac{a+1}{a^k} \leq \frac{(k-1)^{k-1}}{k^k} .$$

If we put  $a = -|a|$  , then (7) follows.

**Lemma 2.2** . Let  $\xi$  be real root of Eq.(6) . The following statements hold true :

(i)  $c$  increases as  $\xi$  increases if either  $0 < \xi < \frac{k}{k-1}$  or  $\xi < 0$  and  $k$  is an odd number .

(ii)  $c$  increases as  $\xi$  decreases if either  $\xi > \frac{k}{k-1}$  or  $\xi < 0$  and  $k$  is an even number .

**Proof.** Since  $\xi$  is a root of Eq. (6), we have  $c = (\xi - 1) / \xi^k$  and

$$\frac{dc}{d\xi} = \xi^{-k-1} [(1-k)\xi + k]. \quad (8)$$

(i) If  $0 < \xi < \frac{k}{k-1}$  or  $\xi < 0$  and  $k$  is an odd number, then  $\frac{dc}{d\xi} > 0$  and  $\xi$  increases with  $c$ .

(ii) In this case we get from (8) that  $\frac{dc}{d\xi} < 0$ . Thus  $\xi$  decreases as  $c$  increases.

**Lemma 2.3** Let  $m \neq n$  be positive integers and  $f(\theta) = \sin m\theta / \sin n\theta$ . Then

(i)  $f(\theta)$  decreases in  $(0, \frac{\pi}{m})$  for  $m > n$  and in  $(0, \pi) \setminus \{0, \frac{\pi}{n}, \frac{2\pi}{n}, \dots, \pi\}$  for  $m = n + 1$ .

(ii)  $f(\theta)$  increases in  $(0, \frac{\pi}{n})$  for  $m < n$  and in  $(0, \pi) \setminus \{0, \frac{\pi}{n}, \frac{2\pi}{n}, \dots, \pi\}$  for  $m = n - 1$ .

**Proof.** We have that

$$f'(\theta) = (m \cos m\theta \sin n\theta - n \cos n\theta \sin m\theta) / \sin^2 n\theta.$$

Letting

$$G(\theta) = 2(m \cos m\theta \sin n\theta - n \cos n\theta \sin m\theta),$$

then

$$G(\theta) = (m - n) \sin(m + n)\theta + (m + n) \sin(m - n)\theta, \quad (9)$$

and

$$G'(\theta) = 2(n^2 - m^2) \sin m\theta \sin n\theta. \quad (10)$$

(i) : Since  $m > n$ , we have that  $G'(\theta) < 0$  for  $0 < \theta < \frac{\pi}{m}$ , but  $G(0) = 0$ . Thus  $f'(\theta) < 0$  in  $(0, \frac{\pi}{m})$  and  $f(\theta)$  is a decreasing function in  $(0, \frac{\pi}{m})$ . Now, let  $m = n + 1$  Eq.(9) becomes

$$G(\theta) = \sin(2n + 1)\theta - (2n + 1) \sin \theta.$$

Since  $G(\theta) < 0$  for  $0 < \theta < \pi$ , it follows that  $f'(\theta) < 0$  and  $f(\theta) = \frac{\sin(n+1)\theta}{\sin n\theta}$  is a decreasing function on  $0 < \theta < \pi$ . The second part (ii) can be proved similarly.



**Lemma 2.4** *Let  $k > 1$  be positive integer, and  $0 < \theta < \pi$ . If  $|a| \geq \frac{k}{k-1}$ , then*

$$\frac{\sin k\theta}{\sin(k-1)\theta} - |a| = 0 \quad (11)$$

*has exactly  $(k-2)$  roots  $\theta_i$  ( $i = 2, 3, \dots, k-1$ ) such that  $\theta_i \in I_i = \left(\frac{(i-1)\pi}{k-1}, \frac{i\pi}{k}\right) = (a_i, b_i)$ . For  $|a| < \frac{k}{k-1}$ , Eq.(11) has an additional root in the interval  $\left(0, \frac{\pi}{k}\right)$ .*

**Proof.** By Lemma 2.3, the function  $f(\theta) = \frac{\sin k\theta}{\sin(k-1)\theta}$  decreases in  $\left(0, \frac{\pi}{k}\right)$ . If  $|a| < \frac{k}{k-1}$ , then from  $\lim_{\theta \rightarrow 0^+} f(\theta) = \frac{k}{k-1}$ , it follows that there exists  $\varepsilon(k) > 0$  such that  $f(\varepsilon) - |a| > 0$  and  $f\left(\frac{\pi}{k}\right) - |a| < 0$ . Therefore, Eq.(11) has only one root in  $\left(0, \frac{\pi}{k}\right)$ . For  $|a| \geq \frac{k}{k-1}$ , it is clear that  $f(0^+) < \frac{k}{k-1} \leq |a|$  and  $f\left(\frac{\pi}{k}\right) < \frac{k}{k-1} \leq |a|$ . Since  $f(\theta)$  decreases in  $\left(0, \frac{\pi}{k}\right)$ , it follows that  $f(\theta) - \frac{k}{k-1} = 0$  has no roots in  $\left(0, \frac{\pi}{k}\right)$ .

Now we show that every interval  $I_i$  ( $i = 2, 3, \dots, k-1$ ) contains only one root of Eq.(11). We notice that for any  $|a|$ , there exists a number  $\tau \in I_i$  such that  $f(\tau) - |a| > 0$  (this follows from  $\lim_{\theta \rightarrow a_i} f(\theta) = \infty$ ), also  $f(b_i) - |a| < 0$ . Therefore,  $I_i$  contains at least one root of equation Eq.(11). To complete the proof, it is enough to show that  $f(\theta)$  decreases in any interval  $I_i$ . Following the steps of the proof of Lemma 2.3, where  $m = k$  and  $n = k-1$ , then Eq.(10) becomes

$$G'(\theta) = -2(2k-1)\sin k\theta \sin(k-1)\theta. \quad (12)$$

In the interval  $I_i$  we have

$$(i-1)\pi < \frac{(i-1)k\pi}{k-1} < k\theta < i\pi$$

and

$$(i-1)\pi < \frac{(i-1)k\pi}{k-1} < k\theta < i\pi$$

and

$$(i-1)\pi < (k-1)\theta < \frac{(k-1)i\pi}{k} < i\pi$$

Therefore,  $\frac{\sin k\theta}{\sin(k-1)\theta} > 0$  in  $I_i$  ( $i = 2, 3, \dots, k-1$ ) and  $G'(\theta) < 0$ . It follows that  $f'(\theta) < 0$  in  $I_i$  and  $f(\theta)$  decreases in  $I_i$  and the root of  $f(\theta) - |a| = 0$  is unique in  $I_i$ .

**Lemma 2.5** *The number of real roots of Eq.(6) is given as follows :*

- (i) *Two ( one ) roots for  $c < 0$  and  $k$  is an even ( odd ) number .*
- (ii) *No roots ( one root ) for  $c > \beta_k = \frac{(k-1)^{k-1}}{k^k}$  and  $k$  is an even ( odd ) number .*
- (iii) *Two ( three ) roots for  $c < \beta_k$  and  $k$  is an even ( odd ) number .*

**Proof.** The validity of this lemma can be shown graphically from the graphs of  $\eta = c\mu^k$  and  $\eta = \mu - 1$ . Note that when  $c = \beta_k$ , the line  $\eta = \mu - 1$  is tangent to the curve  $\eta = c\mu^k$ . The analytical proof will appear in the sequel. If we write complex roots of Eq.(6) in the form  $\mu = r(\cos \theta + i \sin \theta)$ , we get the following equations

$$cr^k \cos k\theta - r \cos \theta + 1 = 0 \quad (13)$$

$$cr^k \sin k\theta - r \sin \theta = 0. \quad (14)$$

From Eqs.(13) and (8), it follows that

$$r = \frac{\sin k\theta}{\sin(k-1)\theta} \quad (15)$$

$$c = \frac{\sin \theta}{r^{k-1} \sin k\theta}. \quad (16)$$

Let  $\theta_i$  ( $i = 1, 2, \dots, k-1$ ) be the solutions of Eq.(11), it follows from (15) and (16) that

$$c = \frac{\sin \theta_i}{|a|^{k-1} \sin k\theta_i}. \quad (17)$$

The number of the complex roots of Eq.(6) corresponding to  $\theta_i$  will be discussed in the following.

**Lemma 2.6** . *The number of complex roots of Eq.(6) equals the number of solutions of Eq.(11) .*

**Proof.** It is clear that if  $\theta_i$  is a solution of Eq.(11) in  $(0, \pi)$  , then  $2\pi - \theta_i$  is also a solution. This means that  $\theta_i$  and  $2\pi - \theta_i$  are corresponding to complex roots of Eq.(6) that are conjugate pairs . Thus if Eq.(14) has  $N$  roots in  $(0, \pi)$  , then it has  $2N$  roots in  $(0, 2\pi)$  .

*Case 1 :  $c < 0$  .* By Lemma 2.2, Eq.(11) has  $k - 1$  roots for  $|a| < \frac{k}{k-1}$  and  $k - 2$  roots for  $|a| > \frac{k}{k-1}$  . We choose those values of  $\theta_i$  such that  $c < 0$  . From Eq.(16) we conclude that  $c < 0$  when  $\sin k\theta_i < 0$  . From Lemma 2.2. we have

$$\frac{(i-1)\pi}{k-1} < \theta_i < \frac{i\pi}{k} ,$$

where  $i = 1, 2, \dots, k-1$  for  $|a| < \frac{k}{k-1}$  and  $i = 2, 3, \dots, k-1$  for  $|a| > \frac{k}{k-1}$  . Hence

$$(i-1)\pi < \frac{k(i-1)\pi}{k-1} < k\theta_i < i\pi . \quad (18)$$

If  $k$  is even , then  $\sin k\theta_i < 0$  for  $i = 2, 4, 6, \dots, k-2$  and the number of roots in  $(0, \pi)$  is  $\frac{k-2}{2}$  and consequently  $k-2$  in  $(0, 2\pi)$  . If  $k$  is odd , then  $\sin k\theta_i < 0$  for  $i = 2, 4, 6, \dots, k-2$  and the number of roots in  $(0, 2\pi)$  is  $k-1$  .

*Case 2* may be proved similarly.

### 3 Main Results

In what follows we give theorems which guarantee that all roots of Eq.(6) are inside the disk  $|\mu| < |a|$  .

**Theorem 3.1** *Let  $k > 1$  be an integer, and  $c$  an arbitrary real . Then all complex roots of Eq.(6) are inside the disk  $|\mu| < |a|$  if and only if*

$$|c| > \frac{(1 + a^2 - 2|a|\cos\phi)^{\frac{1}{2}}}{|a|^k} , \quad (19)$$

where  $\phi$  is the solution in  $I = \left(\frac{(k-2)\pi}{k-1}, \frac{(k-1)\pi}{k}\right)$  of  $\frac{\sin k\theta}{\sin(k-1)\theta} = |a|$  .

**Proof.** The complex roots of (6) ,  $\mu = r(\cos \theta + i \sin \theta)$  ,where  $0 < \theta < 2\pi$  and  $r > 0$  , can be obtained from (13) and (11) . Applying simple operations on (13) and (11) , one obtains the following :

$$|c| = \frac{1}{r^{k-1}} \left| \frac{\sin \theta}{\sin k\theta} \right| .$$

The level curves of  $F(c, r, \theta) = |c| r^{k-1} - \left| \frac{\sin \theta}{\sin k\theta} \right|$  are given by  $\theta = \text{constant}$  . Assume that  $|a| < \frac{k}{k-1}$  ( the case  $|a| > \frac{k}{k-1}$  can be treated similarly) and  $\theta_1 < \theta_2 < \dots < \theta_{k-1}$  are the solutions of  $|a| = \frac{\sin k\theta}{\sin(k-1)\theta}$  in  $(0, \pi)$  . The equations of the level curves in the  $(r, c)$  plane are given by

$$|c(r)| = \frac{1}{r^{k-1}} \left| \frac{\sin \theta_i}{\sin k\theta_i} \right| , \quad i = 1, 2, \dots, k-1 .$$

It is clear that  $|c(r)|$  is a decreasing function. Since  $|c(|a|)| = \frac{(1+a^2-2|a|\cos\theta)^{\frac{1}{2}}}{|a|^k}$  is an increasing function of  $\theta$  in  $(0, \pi)$  ,  $\left(\frac{d|c|}{d\theta} > 0\right)$  , then for  $r = |a|$  and  $\theta_1 < \theta_2 < \dots < \theta_{k-1}$  , we get the corresponding values

$$|c_i(|a|)| = \frac{(1+a^2-2|a|\cos\theta_i)^{\frac{1}{2}}}{|a|^k} , \quad i = 1, 2, \dots, k-1$$

that satisfy  $|c_1| < |c_2| < \dots < |c_{k-1}|$  .

It is not difficult to show that  $r < |a|$  if and only if  $|c| > |c_{k-1}(|a|)|$  . In fact since  $|c(r)|$  is a decreasing function,  $|c(r)| > |c_{k-1}(|a|)|$  implies  $r < |a|$  . Also  $r < |a|$  implies that  $|c(r)| > |c_{k-1}(|a|)|$  , for if we assume the contrary , we obtain  $r \geq |a|$  . This is a contradiction. Since  $|c_{k-1}(|a|)|$  is the corresponding value to  $\theta_{k-1} = \phi$  that lies in  $\left(\frac{(k-2)\pi}{k-1}, \frac{(k-1)\pi}{k}\right)$  , the proof is complete .

**Theorem 3.2** . Let  $k > 1$  be an odd integer and  $c > 0$  . Then all roots of Eq.(6) are inside the disk  $|\mu| < |a|$  if and only if

$$c > \frac{|a| + 1}{|a|^k} \quad (20)$$

**Proof.** First we deal with the roots of Eq.(6) . Set  $F(\mu) = c\mu^k - \mu + 1$  and note that  $F(0) > 0$  ,  $F(-\infty) < 0$  and  $F''(\mu) < 0$  for  $\mu < 0$  it follows that Eq.(6) has one negative root. If  $c = \beta_k = \frac{(k-1)^{k-1}}{k^k}$  , then Eq.(6) has a double root  $\mu = \frac{k}{k-1}$  . Putting  $F_t(\mu) = \beta_k \mu^k - \mu + 1$  , we see that if  $c > \beta_k$  then  $F\left(\frac{k}{k-1}\right) > F_t\left(\frac{k}{k-1}\right) = 0$  , and so Eq.(6) has no positive roots . If  $c < \beta_k$  , then  $F\left(\frac{k}{k-1}\right) < F_t\left(\frac{k}{k-1}\right) = 0$  , and so Eq.(6) has exactly two positive roots .

Now , consider the equation

$$F_a(\mu) \equiv c_a \mu^k - \mu + 1 = 0 \quad (21)$$

where

$$c_a = \frac{|a|+1}{|a|^k}$$

*Case 1 :*  $c_a > \beta_k$ . There are no positive roots and  $c > c_a$  is a sufficient condition.

*Case 2 :*  $c_a < \beta_k$ . For  $c_a < c < \beta_k$  , Eq.(6) has two positive roots  $\xi_1 < \frac{k}{k-1}$  ,  $\xi_2 > \frac{k}{k-1}$  and a negative root  $\xi_3$  . Lemma 2.1. implies that  $|\xi_3| < |a|$  .

To prove that  $\xi_1$  and  $\xi_2$  are less than  $|a|$  , it is enough to show that  $\xi_2 < |a|$  for  $c_a < c < \beta_k$  . For this aim we show that  $F(\mu) > 0$  for  $\mu \geq |a|$  . In fact , for  $\mu \geq |a|$  , we have that  $F'_a(\mu) \geq \frac{k(1+|a|)}{|a|} - 1 > 0$  , and  $F_a(|a|) > 0$  , and so  $F_a(\mu) > 0$  . Since  $F(\mu) > F_a(\mu)$  , then  $F(\mu) > 0$  . Therefore the real roots of Eq.(6) are inside the disk  $|\mu| < |a|$  if and only if  $c > c_a$  . Theorem 3.1 implies that all complex roots of Eq.(6) are inside the unit disk  $|\mu| < |a|$  if and only if

$$|c| > \frac{(1 + a^2 - 2|a| \cos \phi)^{\frac{1}{2}}}{|a|^k} ,$$

where  $\phi$  as defined in Theorem 3.1

**Theorem 3.3 .** Let  $k > 1$  be an odd integer and  $c < 0$  . Then all roots of Eq.(6) are inside the disk  $|\mu| < |a|$  if and only if

$$c < - \frac{(1 + a^2 - 2|a| \cos \phi)^{\frac{1}{2}}}{|a|^k} , \quad (22)$$

where  $\phi$  as defined in Theorem 3.1

**Proof.** Since  $F(\mu) = c\mu^k - \mu + 1$  satisfies the following properties :  $F(0)F(1) < 0$  ,  $F(\mu) < 0$  for  $\mu \geq 1$  ,  $F(\mu) > 0$  for  $\mu \leq 0$  , and  $F'(\mu) < 0$  for  $0 < \mu < 1$  , then Eq.(6) has one positive root  $\xi < 1$  . If  $|a| > 1$  , then the positive root  $\xi < |a|$  . It follows from Theorem 3.1 that all the roots of Eq.(6) are inside the disk  $|\mu| < |a|$  if and only if (12) holds . If  $|a| < 1$  , then the equation

$$\frac{|a| - 1}{|a|^k} \mu^k - \mu + 1 = 0$$

has the positive root  $\mu = |a|$  . Applying Lemma 2.1, we conclude that the positive root  $\xi$  of Eq.(6) satisfies  $\xi < |a|$  if  $c < \frac{|a|-1}{|a|^k}$  . Using Theorem 3.2. and the fact that

$$- \frac{(1 + a^2 - 2|a|\cos\phi)^{\frac{1}{2}}}{|a|^k} < \frac{|a| - 1}{|a|^k}$$

for  $|a| < 1$  , yields the desired result.

**Theorem 3.4 .** *Let  $k > 1$  be an even integer and  $c > 0$  . Then the following statements hold true :*

(i) *if  $c > \frac{(k-1)^{k-1}}{k^k} = \beta_k$  , then all roots of Eq.(6) are inside the disk  $|\mu| < |a|$  if and only if*

$$c > \frac{(1 + a^2 - 2|a|\cos\phi)^{\frac{1}{2}}}{|a|^k} \quad (23)$$

(ii) *if  $|a| > \frac{k}{k-1}$  , then all roots of Eq.(6) are inside the disk  $|\mu| < |a|$  if and only if*

$$\frac{(1 + a^2 - 2a\cos\phi)^{\frac{1}{2}}}{|a|^k} < c < \beta_k \quad (24)$$

where  $\phi$  as defined in Theorem 3.1

**Proof.** (i) If  $c > \beta_k$  , then Eq.(6) has no real roots and Theorem 3.1 implies that all complex roots are inside the disk  $|\mu| < |a|$  if and only if (23) holds .

(ii) If  $c < \beta_k$ , then Eq.(6) has two positive roots  $0 < \xi_1 < \frac{k}{k-1}$  and  $\xi_2 > \frac{k}{k-1}$ . The proof is similar to that of Theorem 3.3. For  $|a| < \frac{k}{k-1}$ , it follows that  $\xi_2 > |a|$ . For  $|a| > \frac{k}{k-1}$ , the equation

$$\frac{|a| - 1}{|a|^k} \mu^k - \mu + 1 = 0$$

has the positive root  $\mu = |a|$ . Applying Lemma 2.1. we conclude that if  $\frac{|a|-1}{|a|^k} < c < \beta_k$ , then  $|a| > \xi_2 > \frac{k}{k-1}$ . Since

$$\frac{|a| - 1}{|a|^k} < \frac{(1 + a^2 - 2|a|\cos\phi)^{\frac{1}{2}}}{|a|^k}.$$

it follows by Theorem 3.1 that all the roots of Eq.(6) are inside the disk  $|\mu| < |a|$  if and only if (23) holds.

**Theorem 3.5** . Let  $k > 1$  be an even integer and  $c < 0$ . Then all roots of Eq.(6) are inside the disk  $|\mu| < |a|$  if and only if

$$c < -\frac{|a| + 1}{|a|^k}$$

**Proof.** For  $F(\mu) = c\mu^k - \mu + 1$ , we notice that  $F(0)F(1) < 0$  and for  $0 \leq \mu \leq 1$  we have  $F'(\mu) = kc\mu^{k-1} - 1 < 0$ . Therefore Eq.(6) has exactly one positive root in  $(0, 1)$ . It is clear that  $F(\mu) < 0$  for  $\mu > 1$ , and so Eq.(6) has no other positive roots. Similarly it is not difficult to show that Eq.(6) has only one negative root. Consider the equation

$$-\frac{|a| + 1}{|a|^k} \mu^k - \mu + 1 = 0$$

that has a negative root  $\mu = -|a|$ . If  $c < -\frac{|a|+1}{|a|^k}$ , then Lemma 1. implies that the negative root  $\xi_1$  satisfies  $|\xi_1| < |a|$ . Clearly that the positive root  $\xi_2 < |a|$  for  $|a| > 1$ . If  $|a| < 1$ , then from  $F(-\infty) < 0$  and  $F(-\xi_2) = c\xi_2^k + 1 + \xi_2 = 2\xi_2 > 0$ , we conclude that  $\xi_1 < -\xi_2$ , hence  $\xi_2 < |\xi_1| < |a|$ . The proof would be complete if we apply Theorem 3.1 to complex roots and observing that

$$-\frac{|a|+1}{|a|^k} < -\frac{(1+a^2-2|a|\cos\phi)^{\frac{1}{2}}}{|a|^k}.$$

Next we present necessary and sufficient conditions for the roots of Eq.(5) to be inside the unit disk .

**Theorem 3.6** . *Let  $k > 1$  be an odd integer . Then all the roots of Eq.(6) are inside the unit disk if and only if one of the following conditions hold.*

(i)  $b > a + 1$  and  $a > 0$  .

(ii)  $b < a - 1$  and  $a < 0$  .

(iii)  $b < -(1+a^2-2a\cos\phi)^{\frac{1}{2}}$  and  $a > 0$  .

(iv)  $b > (1+a^2-2|a|\cos\phi)^{\frac{1}{2}}$  and  $a < 0$  ,

where  $\phi$  is the solution in  $I = \left(\frac{(k-2)\pi}{k-1}, \frac{(k-1)\pi}{k}\right)$  of  $\frac{\sin k\theta}{\sin(k-1)\theta} = |a|$  .

**Proof.** We recall first that the roots of Eq.(5) are inside the unit disk if and only if the roots of Eq.(6) are inside the disk  $|\mu| < |a|$  .

(i) Since  $a > 0$  and  $b > 0$  , then  $c > 0$  and Theorem 3.2. implies that all roots of Eq.(5) are inside the unit disk if and only if

$$c = \frac{b}{a^k} > \frac{a+1}{a^k}$$

i.e.  $b > a + 1$  .

(ii) Here also we have that  $c > 0$  . Theorem 3.2 implies that all roots of Eq.(5) are inside the unit disk if and only if

$$\frac{b}{a^k} > \frac{-a+1}{(-a)^k} = \frac{a-1}{a^k}$$

or  $b < a - 1$  . Note that  $a^k < 0$  .

(iii) Since  $a > 0$  and  $b < 0$  , then  $c < 0$  and Theorem 3.3 implies that all roots of Eq.(5) are inside the unit disk if and only if

$$\frac{b}{a^k} < -\frac{(1+a^2-2a\cos\phi)^{\frac{1}{2}}}{a^k}.$$

(iv) Since  $a < 0$  and  $b > 0$  , then  $c < 0$  and Theorem 3.3. implies that all roots of Eq.(5) are inside the unit disk if and only if



$$\frac{b}{a^k} < - \frac{(1 + a^2 - 2|a| \cos \phi)^{\frac{1}{2}}}{-a^k}$$

or equivalently

$$b > \left(1 + a^2 - 2|a| \cos \phi\right)^{\frac{1}{2}} .$$

**Theorem 3.7 .** *Let  $k > 1$  be an even integer . Then all the roots of Eq.(6) are inside the unit disk if and only if one of the following conditions holds :*

(i)  $b < -|a| - 1$

(ii)  $b > (1 + a^2 - 2|a| \cos \phi)^{\frac{1}{2}}$  and either  $b > \frac{(k-1)^{k-1}}{k^k} a^k$

or  $b < \frac{(k-1)^{k-1}}{k^k} a^k$  and  $|a| > \frac{k}{k-1}$  ,

where  $\phi$  is the solution in  $I = \left(\frac{(k-2)\pi}{k-1}, \frac{(k-1)\pi}{k}\right)$  of  $\frac{\sin k\theta}{\sin(k-1)\theta} = |a|$  .

**Proof .** (i) Since  $b < 0$  , then  $c < 0$  and Theorem 3.5. implies that all roots of Eq.(5) are inside the unit disk if and only if

$$\frac{b}{a^k} < - \frac{|a| + 1}{|a|^k} .$$

The result follows if we notice that  $a^k > 0$ .

(ii) Since  $b > 0$  and  $k$  is an even integer , then  $c = \frac{b}{a^k} > 0$  Applying Theorem 3.5. (i) we obtain that if  $\frac{b}{a^k} > \frac{(k-1)^{k-1}}{k^k}$  , then all roots of Eq.(5) are inside the unit disk if and only if

$$\frac{b}{a^k} > \frac{(1 + a^2 - 2|a| \cos \phi)^{\frac{1}{2}}}{-a^k} .$$

Thus the first part of (iii) follows directly . The second part of (iii) can be proved similarly using Theorem 3.4. (ii) .

Using Theorems 3.6 and 3.7 we have the following fundamental result.

**Theorem 3.8** . Let  $k > 1$  be an integer and let  $a, b$  be arbitrary reals . Equation (3) is asymptotically stable if and only if Conditions (i) - (iv) of Theorem 3.6. holds for even  $k$  or Conditions (i)- (ii) of Theorem 3.7. for odd  $k$  .

Figures 1 and 2 show the domains of  $(a, b)$  for which the roots of  $b\lambda^k - a\lambda + 1 = 0$  with  $a \neq 0$  and  $k > 1$  , are inside the unit disk and consequently for which the difference equation (3) is asymptotically stable . Finally we establish the counterpart of the result of Levin and May for difference equations of advanced type.

**Theorem 3.9** Let  $a=1$  and  $b$  is an arbitrary real number, and  $k > 1$  is a positive integer. Then the zero solution of Eq(3) is asymptotically stable if and only if either  $b > 2$  or

$$b < -2|\cos[(2k-5)\pi/(4k-2)]| \quad (25)$$

**Proof.** From Theorems 3.6 and 3.7, the zero solution of Eq(3) is asymptotically stable if and only if

$$(i) \quad b > 2 \text{ or } b < -\sqrt{2-2\cos\phi} \text{ if } k \text{ is odd.}$$

$$(ii) \quad b < -2 \text{ or } b > \sqrt{2-2\cos\phi} \text{ and } b > \frac{(k-1)^{k-1}}{k^k}$$

where  $\phi$  is the solution in  $I = \left(\frac{(k-2)\pi}{k-1}, \frac{(k-1)\pi}{k}\right)$  of  $\frac{\sin k\theta}{\sin(k-1)\theta} = |a|$  . We first observe that  $b > 2$  implies  $b > \sqrt{2-2\cos(\phi)}$  which in turn implies that  $b > \frac{(k-1)^{k-1}}{k^k}$  . Furthermore,  $b < \sqrt{2-2\cos\phi}$  implies that  $b < -2$ . Hence the zero solution of Eq(3) is asymptotically stable if and only if  $b > 2$  or

$$b < -\sqrt{2-2\cos\phi} = -2\sin(\phi). \quad (26)$$

Since  $\sin(k\phi) = \sin(k-1)\phi$ , it follows that either  $(k-1)\phi + k\phi = (2n+1)\pi$  or  $(k-1)\phi = k\phi + 2n\pi, n \in \mathbb{Z}$ . But the second option is invalid for  $\phi \in I$ . The first option yields  $n = k-2$ . Hence

$$\phi = \left(\frac{(2k-3)\pi}{2k-1}\right). \quad (27)$$

For  $k = 2$ ,  $\phi = \pi/3$  lies in the first quadrant and for  $k > 2$ ,  $\phi$  lies in the second quadrant. Hence

$$-2 \sin(\phi) = -2 \left| \cos \left( \frac{((2k-5)\pi)}{4k-2} \right) \right| \quad (28)$$

which establishes the theorem.

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# The Hilbert Transform of Almost Periodic Functions and Distributions

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## Abstract

Let  $B$  be the space of almost periodic functions in the sense of Bohr defined on the real line metrized by the scalar product defined by the formula

$$(f, g) = M \{f(t)\bar{g}(t)\} = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T f(t)\bar{g}(t) dt$$

and let  $B_{ap}^{ac}$  be a subspace of  $B$  which is closed with respect to the operators  $D$  of differentiation and  $H$  of Hilbert transformation. We equip the space  $B_{ap}^{ac}$  with a locally convex topology so that its dual space  $B_{ap}'^{ac}$  contains the Schwartz space of almost periodic distributions. We establish an inversion formula

$$H^2 f = -f + k \quad \forall f \in B_{ap}'^{ac}$$

where  $k \in B_{ap}'^{ac}$  is an arbitrary constant; we use our results to solve some singular integro-differential equations in the space  $B_{ap}'^{ac}$ . An application of our result to a boundary value problem is also discussed.

## 1 Introduction

**Definition (Bohr) 1.** Let  $f$  be a continuous function defined on  $\mathbb{R}$ . We say that  $f$  is *Bohr almost periodic* on  $\mathbb{R}$  if for any  $\epsilon > 0$  there exists a real number  $l(\epsilon) > 0$  such that for any  $a \in \mathbb{R}$ , there exists a  $\tau \in [a, a + l]$  satisfying  $|f(t + \tau) - f(t)| < \epsilon$  for any  $t \in \mathbb{R}$ .

**Theorem (Bohr) 1.** Let  $L$  be the set of all trigonometric polynomials of the form

$$\sum_{k=1}^{\infty} a_k e^{i\lambda_k t}, \quad t \in \mathbb{R}$$

Adding to  $L$  the limits of the sequences of functions of  $L$  which are uniformly convergent on  $\mathbb{R}$  we get a set  $B$  of continuous functions (see [1], p. 28). A continuous function  $f(t)$  defined on  $\mathbb{R}$  belongs to  $B$  if and only if it is almost periodic on  $\mathbb{R}$ .

The sum of two continuous periodic functions with periods  $p_1$  and  $p_2$  respectively is periodic if and only if there exist non-zero integers  $n_1$  and  $n_2$  such that  $n_1 p_1 = n_2 p_2$ . The function  $a_1 e^{4xi} + a_2 e^{7xi}$  is a periodic function of  $x$  with period  $2\pi$  but the function  $a_1 e^{3xi} + a_2 e^{\sqrt{2}xi}$  is not a periodic function of  $x$ , it is however an almost periodic function of  $x$ . The Weierstrass function

$$\sum_{n=1}^{\infty} b^n \cos(a^n \pi x), \quad 0 < b < 1, \quad ab > 1 + \frac{3\pi}{2}, \quad a \text{ is an odd positive integer},$$

is an example of the almost periodic function which is not differentiable anywhere on the real line whereas the almost periodic function represented by

$$\sum_{n=1}^{\infty} \frac{1}{n^2} e^{\frac{x}{n}}$$

is differentiable everywhere on the real line.

The space  $B$  of the almost periodic functions on  $\mathbb{R}$  is metrized by the norm derived from the inner product defined by

$$(f, g) = M\{f(t)\bar{g}(t)\} = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T f(t)\bar{g}(t) dt$$

$$\|f\| \text{ is defined by } \sqrt{(f, f)}, \quad f \in B \quad (1)$$

The space  $B$  is an inner product space but is not a Hilbert space. This can be easily proved by applying the Riesz representation theorem to the bounded linear functional  $F$  defined on  $B$  by

$$F(\phi) = \lim_{t \rightarrow \infty} \frac{1}{T} \int_0^T \phi(t) dt, \quad \phi \in B \text{ (see [1], pp. 132–138).}$$

It can be seen that  $\tilde{B}$ , the completion of  $B$  with respect to the norm defined by (1), is not separable ([1], p. 29).

## 2 A Useful Theorem

**Theorem 2.** Let  $f(t)$  be an almost periodic function defined on  $\mathbb{R}$ . Then there exists a sequence of real numbers  $\lambda_1, \lambda_2, \lambda_3, \dots$  and the complex constants  $c_1, c_2, c_3, \dots$  called Fourier coefficients of  $f$  such that

$$f(t) = \sum_k^{\infty} c_k e^{\lambda_k t i}$$

$$c_k = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T f(t) e^{-\lambda_k t i} dt$$



the limit in the above sum being interpreted in the mean square sense, i.e., as defined by norm (1)

$$L = \left\| \sum_{k=1}^n c_k e^{\lambda_k t i} - f(t) \right\|^2 = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \left\| \sum_{k=1}^n c_k e^{\lambda_k t i} - f(t) \right\|^2 dt \rightarrow 0 \text{ as } n \rightarrow \infty.$$

For a proof see [1], pp. 133–138.

### 3 Semi-almost periodic distributions

**Definition 2.** The *testing function space*  $B_{ap}^{ac}$  consists of infinitely differentiable a.p. functions  $\phi(t)$  defined on  $\mathbb{R}$  such that  $\phi(t)$  along with all its derivatives is almost periodic and that  $\phi(t)$  has a Fourier expansion

$$\phi(t) = \sum_{m=1}^{\infty} a_m e^{\lambda_m t i}, \quad \lambda_m \text{ real}$$

such that the series

$$\sum_{m=1}^{\infty} |a_m| |\lambda_m|^k$$

is convergent for each  $k = 0, 1, 2, 3, \dots$

Therefore,

$$\phi^{(k)}(t) = \sum_{m=1}^{\infty} a_m (i\lambda_m)^k e^{\lambda_m t i}, \quad k = 0, 1, 2, 3, \dots$$

Thus the space  $B_{ap}^{ac}$  is closed with respect to the operator  $D$  of differentiation. Clearly the almost periodic function

$$\sum_{n=1}^{\infty} \frac{1}{n^2} e^{\frac{x i}{n}}$$

belongs to the space  $B_{ap}^{ac}$ .

Later on in Section 4 we will also see that the space  $B_{ap}^{ac}$  is also closed with respect to the operator  $H$  of Hilbert transformations. The topology over the space  $B_{ap}^{ac}$  is generated by the sequence of seminorms  $\{\gamma_k\}_{k=0}^{\infty}$  where

$$\gamma_k(\phi) = \left( \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |\phi^{(k)}(t)|^2 dt \right)^{\frac{1}{2}}$$

Therefore a sequence  $\{\phi_\nu(t)\}_{\nu=1}^{\infty}$  in  $B_{ap}^{ac}$  converges to 0 in  $B_{ap}^{ac}$  if and only if

$$\gamma_k(\phi_\nu) \rightarrow 0 \text{ as } \nu \rightarrow \infty \text{ for each } k = 0, 1, 2, 3, \dots \quad (\text{see [7], p. 8}).$$

If  $f \in B_{ap}^{'ac}$  there exists a constant  $C > 0$  not depending upon  $\phi$  such that

$$|\langle f, \phi \rangle| \leq C[\gamma_0(\phi) + \gamma_1(\phi) + \dots + \gamma_k(\phi)] \quad (2)$$

We now define the space  $B_{ap}^{(k)ac}$  by

$$B_{ap}^{(k)ac} = \{\psi : \psi = \phi^{(k)}(t), \quad \phi \in B_{ap}^{ac}\}, \quad k = 0, 1, 2, 3, \dots$$

The topology over  $B_{ap}^{(k)ac}$  is generated by the inner product norm as defined by (1). Now in view of (2) and the Hahn–Banach theorem and Riesz representation theorem there exist functions  $g_i$  in the completion of the inner product spaces  $B_{ap}^{(i)ac}$  with respect to the norm defined by (1) such that

$$\begin{aligned}\langle f, \phi \rangle &= \sum_{i=0}^k \langle g_i, \phi^{(i)} \rangle \\ &= \sum_{i=0}^k \langle (-1)^i g_i^{(i)}, \phi_i \rangle \\ &= \left\langle \sum_{i=0}^k (-1)^i g_i^{(i)}, \phi_i \right\rangle \quad \forall \phi \in B_{ap}^{ac}\end{aligned}\tag{3}$$

The space  $B_{ap}^{(k)ac}$ ,  $k = 1, 2, 3, \dots$  is not dense in  $B_{ap}^{ac}$  and so the representation given in (3) is not unique. It may be noted that simple functions belong to the completion of the space  $B_{ap}^{(i)ac}$ ,  $i = 0, 1, 2, 3, \dots$ . In the derivation of the representation formula (3) we have also made use of the fact that the dual of the space  $H_1 \times H_2 \times H_3 \times \dots \times H_k$  is the space  $H_1 \times H_2 \times H_3 \times \dots \times H_k$  where  $H^i$ 's are Hilbert spaces. Let  $f(t)$  be a mean square integrable function on  $\mathbb{R}$  satisfying

$$\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |f(t)|^2 dt < \infty$$

Then a continuous linear functional over the completion  $\tilde{B}_{ap}^{ac}$  of  $B_{ap}^{ac}$  with respect to the norm (1) can be generated by

$$\langle \tilde{f}, \phi \rangle = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T f(t) \phi(t) dt, \quad \forall \phi \in \tilde{B}_{ap}^{ac}\tag{4}$$

A functional like  $\tilde{f}$  defined by (4) can be called a semi-almost periodic distribution. It can be shown that an almost periodic function defined on  $\mathbb{R}$  is a regular almost periodic distribution or a regular semi-almost periodic distribution in view of (4). Thus the Schwartz space of almost periodic distributions is contained in the space  $B_{ap}^{ac}$  (see [5]).

The function  $\phi_n(t) = e^{2\frac{n}{n+1}ti}$  does not go to  $e^{2ti}$  as  $n \rightarrow \infty$  in the topology of the space  $B_{ap}^{ac}$  whereas the function  $\psi_n(t) = \frac{n}{n+1}e^{ti} \rightarrow e^{ti}$  as  $n \rightarrow \infty$  in the topology of the space  $B_{ap}^{ac}$ .

## 4 The Hilbert Transform of Almost Periodic Distributions

In [3] and [4] (Chapter 7) we have discussed the Hilbert transform of periodic distributions and functions. In this section we want to extend the operation of the Hilbert transform to almost periodic functions and to almost periodic distributions. In [3] and [4] our definition of the Hilbert transform of a periodic function with period  $2\tau$  which is  $L^p$  integrable over the interval  $[-\tau, \tau]$  was

$$(Hf)(x) = \frac{1}{\pi} \lim_{N \rightarrow \infty} (P) \int_{-N}^N \frac{f(t)}{x-t} dt \equiv \frac{1}{2\tau} (P) \int_{-\tau}^{\tau} f(x-t) \cot\left(\frac{t\pi}{2\tau}\right) dt$$

It turns out that for  $f \in L_{2\tau}^p$ ,  $p > 1$  the integral

$$\lim_{N \rightarrow \infty} \frac{1}{\pi} \frac{f(t)}{x-t} dt$$

exists a.e.

The question arises: if  $f(t)$  is an almost periodic function on  $\mathbb{R}$  then does this integral exists? In general the answer would be in the negative. Perhaps by appropriate choice of coefficients  $a_k$  and  $\lambda_k$  in the Fourier expansion of  $f$  it can be shown that the above limit does not exist for particular cases of  $f$ .

In the special case of an almost periodic function of type  $e^{i\lambda t}$ , which is actually periodic, the Hilbert transform exists. In fact it can be shown by using the technique of contour integration that

$$\lim_{N \rightarrow \infty} \frac{1}{\pi} \int_{-N}^N \frac{e^{i\lambda t}}{x - t} dt = -i \operatorname{sgn}(\lambda) e^{i\lambda x}$$

where

$$\operatorname{sgn}(\lambda) = \begin{cases} 1 & \text{if } \lambda > 0 \\ 0 & \text{if } \lambda = 0 \\ -1 & \text{if } \lambda < 0 \end{cases}$$

Therefore,

$$H \left( \sum_{n=1}^m a_n e^{\lambda_n t i} \right) = -i \sum_{n=1}^m a_n \operatorname{sgn}(\lambda_n) e^{\lambda_n x i}$$

Note that the trigonometric polynomials are almost periodic functions. Since an almost periodic function is a continuous function which is a uniform limit of the sequence of trigonometric polynomials on  $\mathbb{R}$  which are Hilbert transformable, it may happen that an almost periodic function may be Hilbert transformable in general. Therefore the Hilbert transform of an almost periodic function  $f(t)$  which is the uniform limit of the sequence of trigonometric polynomials

$$f_n(t) = \sum_{m=1}^n a_m e^{\lambda_m t i}$$

may be defined by

$$\begin{aligned} (Hf)(x) &= \lim_{n \rightarrow \infty} H(f_n(t)) \\ &= \sum_{m=1}^{\infty} -a_m i \operatorname{sgn}(\lambda_m) e^{\lambda_m x i} \end{aligned} \tag{5}$$

provided this limit exists.

In fact there is a fairly large class of almost periodic functions for which this limit exists and this class of functions is dense in  $B$ . There is another good reason to adopt this limit as the definition of the Hilbert transform of almost periodic functions. In [8] it has been shown that if  $f$  is a periodic function with period  $2\pi$  and of bounded variation then its Fourier series and the corresponding conjugate Fourier series are related by the Hilbert transform, i.e., if

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \\ g(x) &= \sum_{n=1}^{\infty} (a_n \sin nx - b_n \cos nx) \end{aligned}$$

then

$$\begin{aligned} g(x) &= \frac{1}{2\pi} (P) \int_{-\pi}^{\pi} f(t) \cot \frac{x-t}{2} dt \\ &= \frac{1}{2\pi} (P) \int_{-\pi}^{\pi} f(x-t) \cot \frac{t}{2} dt \end{aligned} \tag{6}$$

In fact it is proved in [8] that the Fourier series of  $f(x)$  converges to  $f(x)$  iff the integral in (6) converges. It is true that if  $f_n$  is the partial sum of the Fourier series on  $f(x)$  and  $g_n(x)$  is the partial sum of the corresponding conjugate Fourier series then  $Hf_n(x) = g_n(x)$  and that

$$\begin{aligned}\lim_{n \rightarrow \infty} g_n(x) &= \lim_{n \rightarrow \infty} Hf_n(x) \\ g(x) &= (Hf)(x)\end{aligned}$$

Therefore, the definition (5) as the definition for the Hilbert transform of almost periodic functions is quite legitimate and we will adopt that definition from now on. With this definition of the Hilbert transform we can see that the class of almost periodic functions  $B_{ap}^{ac}$  is Hilbert transformable. That is, if

$$f = \sum_{n=1}^{\infty} a_n e^{\lambda_n t i} \in B_{ap}^{ac}$$

then

$$Hf = -i \sum_{n=1}^{\infty} a_n (\operatorname{sgn} \lambda_n) e^{\lambda_n t i}$$

It is now a very simple exercise to verify that for this class of almost periodic functions we have

$$D^k Hf = H D^k f, \quad f \in B_{ap}^{ac} \quad (7)$$

In analogy with the definition of the Hilbert transform of distributions given in [3], p. 96 we can define the Hilbert transform  $Hf$  of  $f \in B_{ap}^{'ac}$  by

$$\langle Hf, \phi \rangle = \langle f, -H\phi \rangle \quad \forall \phi \in B_{ap}^{ac}$$

One can verify the inversion formulas  $H^2\phi = -\phi + a_0$  where  $a_0$  is the constant term in the Fourier expansion of  $\phi$ , and

$$H^2 f = -f + \eta \quad (8)$$

where  $\langle \eta, \phi \rangle = \langle f, a_0 \rangle$  if  $f = f_o + \sum_{k=1}^n D^k f_k$  where  $f_n$ 's are a.p. functions then we have  $\langle \eta, \phi \rangle = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T f_o(t) a_o dt$ , which exists, and that  $H$  is a mapping from  $B_{ap}^{'ac}$  into itself. Using (7) one can now show that

$$D^k Hf = H D^k f \quad \forall f \in B_{ao}^{'ac}$$

where  $D$  is the operation of distributional differentiation on  $B_{ap}^{'ac}$  defined by

$$\langle Df, \phi \rangle = \langle f, -D\phi \rangle \quad \forall \phi \in B_{ap}^{ac}, f \in B_{ap}^{'ac} \quad (9)$$

Again,

$$\langle \eta', \phi \rangle = \langle \eta, -\phi' \rangle = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T (f_o(t) \times 0) dt = 0$$

Therefore  $\eta$  is equal to some arbitrary constant element of  $B_{ap}^{'ac}$ , which we shall call  $k$  and so

$$H^2 f = -f + k \quad \forall f \in B_{ap}^{'ac}. \quad (10)$$

Now using the continuity property of the inner product one can show that the right hand side limit of (5) always exists in the weak topology of  $B_{ap}^{'ac}$ .

We now state some useful results.

**Theorem 3.**

(i) In order that a distribution  $T$  be an almost periodic distribution of Schwartz it is necessary and sufficient that  $T$  is a finite sum of finite order distributional derivatives of almost periodic functions (see [5], p. 208).

(ii) Let  $T$  be a Schwartz distribution which is almost periodic then there exist trigonometric series

$$\sum_{n=1}^{\infty} a_n e^{\lambda_n t i}$$

which converge in the weak distributional sense to  $T$  and that the  $k^{\text{th}}$  order distributional derivative of  $T$  is given by

$$D^k T = \sum_{n=1}^{\infty} a_n (\lambda_n i)^k e^{\lambda_n t i}$$

where the convergence in the latter series is interpreted in the weak distributional sense. This result is true irrespective of whether or not the classical function  $T(t)$  representing the series

$$\sum_{n=1}^{\infty} a_n e^{\lambda_n t i}$$

is differentiable in the classical point-wise sense.

(iii) Let  $f$  be a semi-almost periodic distribution, i.e.,  $f \in B_{ap}'^{ac}$ , then there exists a trigonometric series

$$\sum_{n=1}^{\infty} b_n e^{\mu_n t i}$$

which converges to  $f$  in the weak topology of  $B_{ap}'^{ac}$ , i.e.,

(a)

$$\langle f, \phi \rangle = \lim_{m \rightarrow \infty} \left\langle \sum_{n=1}^m b_n e^{\mu_n t i}, \phi \right\rangle \quad \forall \phi \in B_{ap}^{ac}$$

and so

(b)

$$\langle f, \phi \rangle = \lim_{m \rightarrow \infty} \left\langle \sum_{n=1}^m b_n e^{\mu_n t i}, \phi \right\rangle \quad \forall \phi \in \tilde{B}_{ap}^{ac}$$

where  $\tilde{B}_{ap}^{ac}$  is the completion of  $B_{ap}^{ac}$  with respect to the norm defined in (1).

(c) Let  $D^k f$  be the  $k^{\text{th}}$  distributional derivative of  $f \in B_{ap}'^{ac}$  where

$$f = \sum_{n=1}^{\infty} b_n e^{\mu_n t i}$$

then

$$D^k f = \sum_{n=1}^{\infty} b_n (\mu_n i)^k e^{\mu_n t i}$$

It is easy to figure out that (iii) follows from (ii) and Theorem 2 and (ii) follows from (i).

## 5 Applications

We now apply our results to solve some singular integral equations and some boundary value problems in the space of almost periodic functions and distributions as they are very important in applications since most vibrations are not harmonic.

*Example 1.* Solve the following singular integro-differential equations in the space of semi-almost periodic distributions.

$$(i) \quad \frac{dy}{dt} + kHy = 0 \quad (11)$$

$$(ii) \quad \frac{dy}{dt} + kHy = f \quad (12)$$

where  $f$  is an almost periodic distribution and  $H$  is the Hilbert transformation operator as defined before.

*Solution.*

(i) Operating the operator  $kH$  on both sides of (ii) we get

$$\begin{aligned} \frac{d}{dt} k(Hy) + k^2 H^2 y &= 0 \\ \frac{d}{dt} \left( -\frac{dy}{dt} \right) + k^2 (-y + c) &= 0 \end{aligned}$$

where  $c$  is an arbitrary constant element of  $B_{ap}'^{ac}$ .

$$\frac{d^2 y}{dt^2} + k^2 y = k^2 c \quad (13)$$

$y = c$  is a particular solution of (13) Therefore the general solution to (13) is

$$y = A \cos(kt) + B \sin(kt) + c \quad (14)$$

where  $A, B$  are arbitrary constants. It is easy to verify that (14) satisfies (11).

(ii) The solution to the homogeneous integro-differential equation associated with (12) is given by (14). So we now only need to find a particular solution of (12). Let us take

$$y = \sum_{n=1}^{\infty} b_n e^{\lambda_n t i}$$

where  $b_n$ 's are constants as a trial solution of (12) with

$$f = \sum_{n=1}^{\infty} a_n e^{\lambda_n t i}$$

Substituting for  $f$  and  $y$  in (12) we get

$$\sum_{n=1}^{\infty} (\lambda_n i) b_n e^{\lambda_n t i} + k \left( \sum_{n=1}^{\infty} (-i) \operatorname{sgn}(\lambda_n) b_n e^{\lambda_n t i} \right) = \sum_{n=1}^{\infty} a_n e^{\lambda_n t i}$$

Equating the coefficients of  $e^{\lambda_n ti}$  we get

$$b_n i(\lambda_n - k \operatorname{sgn}(\lambda_n)) = a_n$$

Therefore,

$$b_n = \frac{-a_n i}{\lambda_n - k \operatorname{sgn}(\lambda_n)}$$

Therefore the solution to (12) is

$$y = A \cos(kt) + B \sin(kt) + c + \sum_{n=1}^{\infty} \frac{-a_n i}{\lambda_n - k \operatorname{sgn}(\lambda_n)} e^{\lambda_n ti}$$

It is assumed that none of  $\lambda_i$  assumes the values  $k$  or  $-k$ . If  $\lambda_l = k$  and  $\lambda_j = -k$  then the general solution of (12) will be

$$y = A \cos(kt) + B \sin(kt) + c + \frac{ta_l}{2ki} e^{kti} - \frac{ta_j}{2ki} e^{-ktj} + \sum_{\substack{n=1 \\ n \neq i, j}}^{\infty} \frac{-a_n i}{\lambda_n - k \operatorname{sgn}(\lambda_n)} e^{\lambda_n ti}$$

which is not an almost periodic function or distribution.

*Example 2.* Find the general solution of the singular integro-differential equation

$$y'' + Hy = f \quad (15)$$

in the space  $B'_{ap}$ . It is assumed that  $f \in B'^{ac}_{ap}$ , and  $H$  is the Hilbert transformation operator.

*Solution.* We first find the general solution of the associated homogeneous differential equation

$$y'' + Hy = 0 \quad (16)$$

Take  $y = e^{\lambda ti}$  as a trial solution of (16). Therefore,

$$(\lambda i)^2 e^{\lambda ti} - i \operatorname{sgn}(\lambda) e^{\lambda ti} = 0$$

Therefore,

$$-\lambda^2 - i \operatorname{sgn}(\lambda) = 0$$

But  $\lambda$  is supposed to be real. Therefore there is no solution of the form  $e^{\lambda ti}$  where  $\lambda$  is real and non-zero. One can however verify that  $y = k$ , where  $k$  is any constant is a solution to (16). Now assume that

$$f = \sum_{n=1}^{\infty} a_n e^{\lambda_n ti}$$

and so the solution to (15) is of the form

$$y = \sum_{n=1}^{\infty} b_n e^{\lambda_n ti}$$

Therefore, substituting for  $y$  and  $f$  in (15) and equating the coefficients of  $e^{\lambda_n ti}$  we get

$$b_n = \frac{-a_n}{\lambda_n^2 + i \operatorname{sgn}(\lambda_n)}, \quad n = 1, 2, 3, \dots$$

So a particular solution of (15) is

$$y = \sum_{n=1}^{\infty} \frac{-a_n}{\lambda_n^2 + i \operatorname{sgn}(\lambda_n)} e^{\lambda_n ti}$$

Therefore the general solution to (15) is

$$y = k + \sum_{n=1}^{\infty} \frac{-a_n}{\lambda_n^2 + i \operatorname{sgn}(\lambda_n)} e^{\lambda_n ti}$$

where  $k$  is an arbitrary constant.

*Open Problems.* **It is not yet known how to solve differential equations of the type**

$$\frac{dy}{dt} + a(t)Hy = f(t)$$

**where**

- (i)  $a(t)$  and  $f(t)$  are almost periodic functions.
- (ii)  $a(t)$  and  $f(t)$  are almost periodic distributions in the sense of Schwartz
- (iii)  $a(t)$  and  $f(t)$  are semi-almost periodic distributions belonging to  $B_{ap}^{\prime ac}$ .

More complex problems in higher order differential equations can be formulated. The problems will be still more complex when  $y$  is an  $n$ -dimensional column vector and  $a(t)$  is an  $n \times n$  matrix with almost periodic functions and distributions as its elements and  $f$  is an  $n$ -dimensional column vector consisting of almost periodic functions or distributions. These problems may be discussed in our later publications.

*Example 3.* Find a (harmonic) function  $U(x, y)$  such that for a fixed  $y > 0$ ,  $U(x, y)$  is a finite linear combination of finite order derivatives of a.p. functions and

$$U_{xx} + U_{yy} = 0, \quad x \in \mathbb{R}, y > 0$$

$$U(x, y) = O(1), \quad y \rightarrow \infty \text{ uniformly } \forall x \in \mathbb{R}$$

and

$$\lim_{y \rightarrow 0^+} U(x, y) = T$$

in the weak distributional sense where  $T$  is an a.p. distribution of Schwartz having the structure formula

$$T = \sum_{k=0}^n D^k f_k(t), f_k \in B \tag{17}$$



*Solution.* The required solution is

$$\begin{aligned} U(x, y) &= \frac{1}{\pi} \sum_{k=0}^n D_x^k \left\langle f_k(t), \frac{y}{(x-t)^2 + y^2} \right\rangle \\ &= \frac{1}{\pi} \sum_{k=0}^n \int_{-\infty}^{\infty} f_k(t) D_x^k \left[ \frac{y}{(x-t)^2 + y^2} \right] dt \end{aligned} \quad (18)$$

Now for  $\phi \in D$  we have

$$\langle U(x, y), \phi(x) \rangle = \sum_{k=0}^n \frac{1}{\pi} \int_{-\infty}^{\infty} \left\langle f_k, \frac{y}{(x-t)^2 + y^2} \right\rangle (-1)^k \phi^{(k)}(x) dx \quad (19)$$

It is a simple exercise to show that

$$\frac{1}{\pi} \left\langle f_k, \frac{y}{(x-t)^2 + y^2} \right\rangle = \frac{1}{\pi} \int_{-\infty}^{\infty} f_k(t) \frac{y}{(x-t)^2 + y^2} dt \rightarrow f_k \quad (20)$$

uniformly  $\forall x \in \mathbb{R}$  as  $y \rightarrow 0^+$ . Therefore, from (11), (12) and (13) we have

$$\lim_{y \rightarrow 0^+} \langle U(x, y), \phi(x) \rangle = \sum_{k=0}^n \left\langle f_k, (-1)^k \phi^{(k)}(x) \right\rangle \quad (21)$$

$$\begin{aligned} &= \left\langle \sum_{k=0}^n D^k f_k, \phi \right\rangle \\ &= \langle T, \phi \rangle \end{aligned} \quad (22)$$

Since  $f_k(t)$ 's are uniformly bounded on  $\mathbb{R}$  it follows from (12) that

$$U(x, y) = O(1), \quad y \rightarrow \infty$$

uniformly for all  $x \in \mathbb{R}$ .

These results will be a little harder to justify if  $T \in B'_{ap}{}^{ac}$  but is not an almost periodic distribution of Schwartz. In proving these assertions one has also to make use of the fact that the space of trigonometric polynomials is dense in  $B$ .

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# On the convergence of Fourier means and interpolation means

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## Abstract

The necessary and sufficient conditions of the convergence of the Fourier means and the interpolation means in  $L_p$  spaces of  $2\pi$ -periodic functions with  $1 \leq p \leq +\infty$  are established in terms of the Fourier transform of the kernel generator. It is shown that both methods are equivalent from the point of view of their approximation errors. Some applications of the results obtained are given.

**Keywords:** Fourier means, interpolation, approximation errors, necessary and sufficient conditions of convergence

## 1. INTRODUCTION

In the present paper we deal with two classical methods of trigonometric approximation of  $2\pi$ -periodic real valued functions of  $d$  variables: the Fourier means and the interpolation means. Before we discuss briefly the history of the problem and formulate the main purposes of this paper we give exact definitions.

By  $\mathcal{K}$  we denote the class of real valued centrally symmetric functions  $\varphi$  having a compact support with  $\varphi(0) = 1$  which are integrable in the Riemannian sense and are either continuous or essentially discontinuous, that is, do not coincide almost everywhere with any continuous function. Any function in  $\mathcal{K}$  generates the sequence of kernels which is given by ( $\nu x = \nu_1 x_1 + \dots + \nu_d x_d$ )

$$W_n(\varphi)(h) = \sum_{\nu \in \mathbb{Z}^d} \varphi\left(\frac{\nu}{n}\right) \cdot e^{i\nu x}, \quad n \in \mathbb{N}. \quad (1)$$

Clearly, the function  $W_n(\varphi)(h)$  belongs to the space  $\mathcal{T}_N$  of real valued trigonometric polynomials of order at most  $N$  which is defined by

$$\mathcal{T}_N = \left\{ t(x) = \sum_{k \in \mathbb{Z}^d} c_k e^{ikx} : c_{-k} = \bar{c}_k, \quad |k| = (k_1^2 + \dots + k_d^2)^{1/2} \leq N \right\}.$$

with

$$N \equiv N(n; \varphi; \sigma) = [n\sigma] , \quad (2)$$

where  $\sigma$  is an arbitrary real number satisfying

$$\sigma \geq \sigma_\varphi \equiv \sup \left\{ |\xi| = (\xi_1^2 + \dots + \xi_d^2)^{1/2} : \xi \in \text{supp} \varphi \right\} . \quad (3)$$

The Fourier means and the interpolation means are given by

$$\mathcal{F}_n^\varphi(f; x) = (2\pi)^{-d} \int_{\mathbb{T}^d} f(h) \cdot W_n(\varphi)(x - h) dh , \quad n \in \mathbb{N} ; \quad (4)$$

$$\mathcal{I}_n^\varphi(f, x) = (2N + 1)^{-d} \cdot \sum_{k=0}^{2N} f(t_N^k) \cdot W_n(\varphi)(x - t_N^k) , \quad n \in \mathbb{N} \quad (5)$$

respectively. In formulas (4)-(5)  $x, h, k$  are  $d$ -dimensional vectors,  $dh = dh_1 \dots dh_d$ ,

$$t_N^k = \frac{2\pi k}{2N + 1}, \quad k \in \mathbb{Z}^d; \quad \mathbb{T}^d = [0, 2\pi]^d ; \quad \sum_{k=0}^{2N} \equiv \sum_{k_1=0}^{2N} \dots \sum_{k_d=0}^{2N} .$$

Henceforth, we reserve the symbols  $L_p$  and  $\|\cdot\|_p$  for the corresponding space of  $2\pi$ -periodic functions and its standard norm. In the case  $p = +\infty$  they are replaced with the space  $C$  of real valued  $2\pi$ -periodic continuous functions and with  $\|\cdot\|$  for the corresponding norm. If we deal with the  $L_p$ -space on  $\mathbb{R}^d$ , we use the notations  $L_p(\mathbb{R}^d)$  and  $\|\cdot\|_{L_p(\mathbb{R}^d)}$  respectively.

As usual, we say that the means  $\mathcal{F}_n^\varphi$  (the interpolation means  $\mathcal{I}_n^\varphi$ ) have the convergence property in  $L_p$ ,  $1 \leq p \leq +\infty$  (in  $C$ ), if for each  $f \in L_p$  ( $f \in C$ )

$$\lim_{n \rightarrow +\infty} \|f - \mathcal{F}_n^\varphi(f)\|_p = 0 \quad \left( \lim_{n \rightarrow +\infty} \|f - \mathcal{I}_n^\varphi(f)\| = 0 \right) .$$

An enormous number of books and papers is devoted to the method of approximation by the Fourier means. The classical approach deals with concrete methods like the Fourier partial sums or the means of Fejer, Rogosinski, de la Vallee-Poussin, Favard and many others. The general criterion of convergence in the space  $C$  of continuous  $2\pi$ -periodic functions can be obtained as a consequence of the Banach-Steinhaus theorem. It implies some sufficient conditions in terms of smoothness of the function generating the kernel or in terms of its Fourier transform (for references and the history of the problem see, for instance, [4]).

One of the aims of our investigation is to show that if the generator of the kernel of the method has a priori some natural properties, the condition of belonging of its Fourier transform to the space  $L_1(\mathbb{R}^d)$  is not only sufficient but

also necessary for the convergence of the Fourier means in the spaces  $C$ ,  $L_1$  or in  $L_p$  for all  $1 \leq p \leq +\infty$ . We will also show that the condition of belonging of the Fourier transform of the kernel generator to  $L_{\tilde{p}}(\mathbb{R}^d)$ , where  $\tilde{p}$  is equal to  $p$  for  $1 < p \leq 2$  and is its conjugate for  $2 < p < +\infty$ , turns out to be necessary for the convergence in the space  $L_p$  with  $1 < p < +\infty$ . We will also see that this condition can not be improved in some sense. Our conclusions are based on the exact formula for the  $L_1$ -norm of the kernel in terms of the Fourier transform. The same formula will enable us to establish a simple and effective criterion of positivity of the method.

As is well-known, the interpolation means and the corresponding Fourier means very often have the same convergence properties (see, for instance, [15, Vol. 2, Chapter 10]). We will show that the convergence criteria for the interpolation means and for the Fourier means coincide that will explain this phenomenon. The criteria in combination with the fact that the Fourier means and their interpolation counterparts transform a trigonometric polynomial of the order that is strongly connected with the order of the means in accordance with one and the same law will enable us to prove that the approximation errors of both methods are equivalent to each other, that is

$$\|f - \mathcal{I}_n^\varphi(f)\| \asymp \|f - \mathcal{F}_n^\varphi(f)\|, \quad f \in C. \quad (6)$$

Here and throughout the paper we write  $A(f; n; p) \preceq B(f; n; p)$  for quantities depending on  $f \in L_p$ ,  $n \in \mathbb{N}$  and the parameter  $0 < p \leq +\infty$ , if there exists a positive constant  $C \equiv C(p)$ , such that  $A(f; n; p) \leq C \cdot B(f; n; p)$  for all  $f \in L_p$ ,  $n \in \mathbb{N}$ . As usual,  $A(f; n; p) \asymp B(f; n; p)$ , if  $A(f; n; p) \preceq B(f; n; p)$  and  $B(f; n; p) \preceq A(f; n; p)$  simultaneously. For the non-periodic one-dimensional case equivalence (6) can be found in [3].

This result shows that the interpolation means do not require any special investigations. All facts concerning the rate of convergence can be immediately transferred from the case of the Fourier means. Only the constants in the estimates of the approximation errors can be affected.

The paper is organized as follows. In Section 2 we give some well-known facts on the Fourier means and the interpolation means. In Section 3 we deduce the exact formula for the  $L_1$ -norm of the kernel in terms of the Fourier transform of its generator and we establish the criterion of positivity of the Fourier means. Section 4 is devoted to the proofs of the convergence criteria and the equivalence (6).

## 2. PRELIMINARIES

For the sake of convenience of the reader we give in this Section some well-known results on the convergence of the methods (4) and (5) that are based on

the Banach-Steinhaus theorem For more details we refer to [4, Chapter 1]. As usual, the norm of a linear operator  $\mathcal{L}$  that maps  $L_p$ ,  $1 \leq p \leq +\infty$  into itself is given by

$$\|\mathcal{L}\|_{(p)} = \sup_{\|f\|_p \leq 1} \|\mathcal{L}f\|_p .$$

In this Section we will omit the lower indices when denoting the norms of functions and operators in  $C$  as well as the symbol  $\varphi$  in  $\mathcal{F}_n^\varphi$ ,  $\mathcal{I}_n^\varphi$  and  $W_n(\varphi)$ .

### A. Fourier means

As is known (see, for instance, [4, pp. 30-32])

$$\|\mathcal{F}_n\|_{(p)} \leq \|\mathcal{F}_n\|_{(1)} = \|\mathcal{F}_n\|_{(\infty)} = (2\pi)^{-d} \cdot \|W_n\|_1 , \quad n \in \mathbb{N} . \quad (7)$$

Moreover, for each  $m \in \mathbb{Z}^d$ ,  $n \in \mathbb{N}$  we get from (1) and (4)

$$\mathcal{F}_n(e^{im\cdot}; x) = \sum_{\nu \in \mathbb{Z}^d} \varphi\left(\frac{\nu}{n}\right) e^{imx} \cdot (2\pi)^{-d} \int_{\mathbb{T}^d} e^{imh} e^{i\nu h} dh = \varphi\left(-\frac{m}{n}\right) e^{imx} = \varphi\left(\frac{m}{n}\right) e^{imx} . \quad (8)$$

The combination of the Banach-Steinhaus theorem with (7) and (8) gives

**Lemma 1.** *The following statements are equivalent:*

- 1)  $\{\mathcal{F}_n\}$  has the convergence property in  $C$  ;
- 2)  $\{\mathcal{F}_n\}$  has the convergence property in  $L_1$  ;
- 3)  $\{\mathcal{F}_n\}$  has the convergence property in  $L_p$  for all  $1 \leq p \leq +\infty$ ;
- 4) (a)  $\lim_{n \rightarrow +\infty} \varphi\left(\frac{m}{n}\right) = 1$  for each  $m \in \mathbb{Z}^d$  and (b) the sequence  $\{\|W_n\|_1\}$  is bounded.

We also notice that in view of (8)

$$\mathcal{F}_n(T; x) = \sum_{k \in \mathbb{Z}^d} \varphi\left(\frac{k}{n}\right) \cdot c_k e^{ikx}, \quad \left( T(x) = \sum_{k \in \mathbb{Z}^d} c_k e^{ikx} \in \mathcal{T}, n \in \mathbb{N} \right), \quad (9)$$

where  $\mathcal{T}$  is the space of all real valued trigonometric polynomials of  $d$  variables.

### B. Interpolation means

First we notice that

$$\|\mathcal{I}_n\| = \max_{x \in \mathbb{T}^d} (2N+1)^{-d} \cdot \sum_{k=0}^{2N} |W_n(x - t_N^k)| , \quad n \in \mathbb{N} . \quad (10)$$

The upper estimate follows immediately from (5). To prove the lower estimate it is enough to select  $x_0$  giving the maxima on the right-hand side of (10) and to consider the  $2\pi$ -periodic continuous function  $f(x)$  that satisfies

$$f(t_N^k) = \operatorname{sgn} W_n(x_0 - t_N^k) \quad , \quad (|k|_\infty \equiv \max\{|k|_1, \dots, |k|_d\} \leq 2N); \quad \|f\| \leq 1 \quad .$$

Now we prove that

$$\|\mathcal{I}_n\| \asymp \|W_n\|_1 \quad . \quad (11)$$

Indeed, we get from (10) by integration

$$\|\mathcal{I}_n\| \geq (2\pi)^{-d} \int_{\mathbb{T}^d} (2N+1)^{-d} \cdot \sum_{k=0}^{2N} |W_n(x - t_N^k)| \, dx = (2\pi)^{-d} \cdot \|W_n\|_1 \quad .$$

The inverse estimate follows from the Marcinkiewicz theorem on the relations between the discrete and the continuous norms of a trigonometric polynomial ([15, Vol. 2, Ch. 10, p. 28]); for the multivariate version see also [7]). Namely, since  $W_n(x - h)$  as a function of  $h$  is a trigonometric polynomial of order at most  $N$  with respect to each component of  $h$ , we get by applying the upper estimate for the discrete norm (which in difference to the lower estimate is valid also for  $p = 1$ )

$$\|\mathcal{I}_n\| \leq c \cdot \max_{x \in \mathbb{T}^d} \|W_n(x - \cdot)\|_1 = c \cdot \|W_n\|_1 \quad .$$

The proof of (11) is complete. As it follows from the proof given in [15, Vol. 2, Ch. 10, p. 28], the last inequality is valid with the constant  $c = 3^d(2\pi)^{-d}$ . Thus,

$$\|W_n\|_1 \leq (2\pi)^d \cdot \|\mathcal{I}_n\| \leq 3^d \cdot \|W_n\|_1 \quad , \quad n \in \mathbb{N} \quad . \quad (12)$$

Taking into account that the inequality  $|\nu| > N$  implies that  $\frac{\nu}{n} \notin \operatorname{supp} \varphi$ , we get for each  $n \in \mathbb{N}$  and  $m \in \mathbb{Z}^d$  satisfying  $|m| \leq N$

$$\begin{aligned} \mathcal{I}_n(e^{im\cdot}; x) &= \sum_{\nu \in \mathbb{Z}^d} \varphi\left(\frac{\nu}{n}\right) e^{i\nu x} \cdot (2N+1)^{-d} \cdot \sum_{k=0}^{2N} \exp\{it_N^k(m - \nu)\} = \\ &= \sum_{\nu: |\nu| \leq N} \varphi\left(\frac{\nu}{n}\right) e^{i\nu x} \cdot \prod_{j=1}^d \delta(m_j; \nu_j) \quad , \end{aligned} \quad (13)$$

where

$$\delta(m_j; \nu_j) = \begin{cases} 1 & , \quad m_j \equiv \nu_j \pmod{2N+1} \\ 0 & , \quad \text{otherwise} \end{cases} \quad , \quad j = 1, \dots, d .$$

Since  $|m_j| + |\nu_j| \leq |m| + |\nu| \leq 2N < 2N+1$ ,  $\delta(m_j; \nu_j) = 1$  for all  $j = 1, \dots, d$  if and

only if  $m = \nu$ ; therefore, we get finally from (13)

$$\mathcal{I}_n(e^{im}; x) = \varphi\left(\frac{m}{n}\right) e^{imx} \quad , \quad |m| \leq N . \quad (14)$$

Applying the Banach-Steinhaus theorem, (11) and (14) we obtain the following

**Lemma 2.** *The following statements are equivalent:*

- 1)  $\{\mathcal{I}_n\}$  has the convergence property in  $C$  ;
- 2) (a)  $\lim_{n \rightarrow +\infty} \varphi\left(\frac{m}{n}\right) = 1$  for each  $m \in \mathbb{Z}^d$  and (b) the sequence  $\{\|W_n\|_1\}$  is bounded.

We also notice that in view of (14)

$$\mathcal{I}_n(T; x) = \sum_{k \in \mathbb{Z}^d} \varphi\left(\frac{k}{n}\right) \cdot c_k e^{ikx} , \quad \left( T(x) = \sum_{k \in \mathbb{Z}^d} c_k e^{ikx} \in \mathcal{T}_N , n \in \mathbb{N} \right) . \quad (15)$$

### 3. THE NORMS OF THE KERNELS

The Fourier transform and its inverse are given for functions  $g \in L_1(\mathbb{R}^d)$  by

$$\begin{aligned} \widehat{g}(x) &= \int_{\mathbb{R}^d} g(\xi) e^{-ix\xi} d\xi ; \\ g^\vee(x) &= (2\pi)^{-d} \int_{\mathbb{R}^d} g(\xi) e^{ix\xi} d\xi . \end{aligned}$$

Before we formulate and prove the main results of this Section we give some remarks on the Poisson summation formula which will be essentially used here. Let a function  $g(\xi)$  be defined everywhere on  $\mathbb{R}^d$  and belong to  $L_1(\mathbb{R}^d)$ . The equation

$$\sum_{k \in \mathbb{Z}^d} g(k) = \sum_{\nu \in \mathbb{Z}^d} \widehat{g}(2\pi\nu) \quad (16)$$



is called the Poisson summation formula. As is well-known, (16) is valid if  $g$  satisfies some additional conditions. If, for example,  $g$  is continuous and

$$|g(\xi)| \leq A(1 + |\xi|)^{-d+\varepsilon} ; \quad |\widehat{g}(x)| \leq A(1 + |x|)^{-d+\varepsilon} \quad (17)$$

for some  $\varepsilon > 0$ , (16) holds ([12, Chapter 7, Corollary 2.6, p. 252]).

We will often use the following version of the Poisson summation formula

$$\sum_{k \in \mathbb{Z}^d} g(k) \cdot e^{ikx} = \sum_{\nu \in \mathbb{Z}^d} \widehat{g}(x + 2\pi\nu) \quad (18)$$

that immediately follows from (16). As is proved in [12, p. 252], it is valid everywhere in  $\mathbb{R}^d$ , if the conditions (17) are satisfied.

For our purposes we need formula (18) with some other conditions on a function  $g$ .

**Lemma 3.** *Suppose that a function  $g(\xi)$  is continuous, it has a compact support and  $\widehat{g}(x) \in L_1(\mathbb{R}^d)$ . Then (18) holds almost everywhere in  $\mathbb{R}^d$  and, moreover,*

$$\left\| \sum_{k \in \mathbb{Z}^d} g(k) \cdot e^{ikx} \right\|_1 \leq \|\widehat{g}\|_{L_1(\mathbb{R}^d)} . \quad (19)$$

**Proof.** The validity of (18) for almost every  $x$  in the one-dimensional case was established in [1, Lemma 2.2]. One can also notice that the proof given there fits for any dimension. To establish (19) we write

$$\left\| \sum_{k \in \mathbb{Z}^d} g(k) \cdot e^{ikx} \right\|_1 \leq \sum_{k \in \mathbb{Z}^d} \|\widehat{g}(x + 2\pi\nu)\|_1 = \|\widehat{g}\|_{L_1(\mathbb{R}^d)}$$

that finishes the proof. ■

**Lemma 4.** *Let  $\varphi(\xi) \in \mathcal{K}$  be continuous. If  $\widehat{\varphi} \in L_1(\mathbb{R}^d)$ , then*

$$\|W_n(\varphi)\|_1 \leq \|\widehat{\varphi}\|_{L_1(\mathbb{R}^d)} , \quad n \in \mathbb{N} , \quad (20)$$

where the functions  $W_n(\varphi)$  are given by (1).

**Proof.** To prove (20) it is enough to apply (19) to the function  $\varphi\left(\frac{\xi}{n}\right)$  and to use the fact that  $\widehat{\varphi(\delta \cdot)}(x) = \delta^{-d} \widehat{\varphi}(\delta^{-1}x)$ . ■

The idea of using the Poisson summation formula for the estimates of type (20) can be found in [5], where the case of infinitely differentiable generators  $\varphi$  was considered.

Lemma 4 establishes a sufficient condition of the boundedness of the sequence of the norms  $\|W_n(\varphi)\|_1$ . Now we prove that the same condition turns out to be necessary.

**Lemma 5.** *Let  $\varphi(\xi) \in \mathcal{K}$  and  $1 \leq p < +\infty$ . If there is a positive constant  $C(p; \varphi)$ , such that*

$$n^{d(1/p-1)} \cdot \|W_n(\varphi)\|_p \leq C(p; \varphi) \quad (21)$$

for all  $n \in \mathbb{N}$ , then  $\widehat{\varphi}(x) \in L_p(\mathbb{R}^d)$  and  $\|\widehat{\varphi}\|_{L_p(\mathbb{R}^d)} \leq C(p; \varphi)$ .

The statement remains valid if (21) is fulfilled only for some strongly increasing sequence  $\{n_k\}_{k=1}^{+\infty} \subset \mathbb{N}$ .

**Proof.** We consider the sequence of functions  $\{F_n(x)\}_{n=1}^{+\infty}$  given by

$$F_n(x) = \begin{cases} n^{-dp} \cdot \left| W_n(\varphi) \left( \frac{x}{n} \right) \right|^p, & x \in [-\pi n, \pi n]^d \\ 0, & \text{otherwise} \end{cases}. \quad (22)$$

Clearly, the functions  $F_n(x)$ ,  $n \in \mathbb{N}$ , are non-negative and measurable. Let  $x_0 \in \mathbb{R}^d$ . Then there exists  $n_0 \in \mathbb{N}$ , such that  $x_0 \in [-\pi n, \pi n]^d$  for  $n \geq n_0$ . The function  $\varphi(\xi)e^{i\xi x_0}$  of the variable  $\xi$  is integrable in the Riemannian sense on the cube  $\Omega \subset \mathbb{R}^d$  which contains its support. By definition of the Riemann integral we get

$$\lim_{n \rightarrow +\infty} n^{-d} \cdot \sum_{k \in \mathbb{Z}^d} \varphi \left( \frac{k}{n} \right) \cdot e^{i \frac{k}{n} x_0} = \int_{\Omega} \varphi(\xi) \cdot e^{i \xi x_0} d\xi = \widehat{\varphi}(-x_0).$$

Therefore,

$$\lim_{n \rightarrow +\infty} F_n(x_0) = |\widehat{\varphi}(-x_0)|^p. \quad (23)$$

>From (21) we get

$$\int_{\mathbb{R}^d} F_n(x) dx = n^{-dp} \cdot \int_{[-\pi n, \pi n]^d} \left| W_n(\varphi) \left( \frac{x}{n} \right) \right|^p dx = n^{d(1-p)} \cdot \|W_n(\varphi)\|_p^p \leq (C(p; \varphi))^p.$$

Thus, we have proved that the sequence  $\{F_n(x)\}_{n=1}^{+\infty}$  satisfies all conditions of the Fatou lemma; therefore, the integral of its limit can be estimated by the same constant, that is,  $\widehat{\varphi} \in L_p(\mathbb{R}^d)$  and  $\|\widehat{\varphi}\|_{L_p(\mathbb{R}^d)} \leq C(p; \varphi)$ .

Replacing the parameter  $n \in \mathbb{N}$  in the proof above with  $n_k$  we obtain the second part of Lemma 5. ■

We notice that the idea of representing formulas like (1) as a Riemann integral sum for the integral which defines the Fourier transform is not new in the case  $p = 1$ . It goes back to [2], [3], [14], where it was applied to some problems of trigonometric approximation as well as to sampling in the space  $C$ .

**Lemma 6.** *If  $\varphi(\xi) \in \mathcal{K}$  is essentially discontinuous, then the sequence  $\{\|W_n(\varphi)\|\}_{n=1}^{+\infty}$  is unbounded.*

**Proof.** If  $\{\|W_n(\varphi)\|\}_{n=1}^{+\infty}$  is bounded, then by virtue of Lemma 5  $\widehat{\varphi} \in L_1(\mathbb{R}^d)$ . Therefore,  $(\widehat{\varphi})^\vee(x) = \varphi(x)$  for almost all  $x \in \mathbb{R}^d$ . This statement can be obtained, for example, by using the Plancherel theorem on the Fourier transform in  $L_2(\mathbb{R}^d)$ . Since  $(\widehat{\varphi})^\vee(x)$  is continuous,  $\varphi(x)$  is also continuous by the definition of the class  $\mathcal{K}$ . ■

Now we formulate the main result of this Section.

**Theorem 1.** *Let  $\varphi(\xi) \in \mathcal{K}$ . Then the sequence  $\{\|W_n(\varphi)\|_1\}_{n=1}^{+\infty}$  is bounded if and only if  $\widehat{\varphi}(x) \in L_1(\mathbb{R}^d)$ . Moreover, in this case it has a limit and*

$$\lim_{n \rightarrow +\infty} \|W_n(\varphi)\|_1 = \sup_{n \in \mathbb{N}} \|W_n(\varphi)\|_1 = \|\widehat{\varphi}\|_{L_1(\mathbb{R}^d)}. \quad (24)$$

**Proof.** Theorem 1 follows immediately from Lemmas 4 - 6. ■

The second equality in (24) (with the supremum) can be also derived from the results on the connections between periodic and non-periodic multipliers [12, Theorem 3.8, p. 260; Theorem 3.18, p.264; Corollary 3.28, p. 267] in combination with the criterion of non-periodic multipliers in  $L_1$  [10, pp. 28, 95].

Theorem 1 gives an explicit formula for the  $L_1$ -norms of the kernels. The fact of existence of the limit seems to be interesting, because as far as we know the previous works on these matters dealt with a suprema only. We also notice that the common methods of treating kernels were mainly based on finding such representation of a given kernel that contained an independent on  $n$  number of items in difference to the sum of type (1). Clearly, such approach is very individual, that is, it essentially depends on what function  $\varphi$  is. The Fourier transform can also be calculated only in rare cases, but, on the other hand, the problem to check whether the Fourier transform of a given function belongs to  $L_1(\mathbb{R}^d)$  can be solved rather than the representation of a kernel described above can be explicitly found. Moreover, when operating with the Fourier transform we do not have a parameter  $n$  any more in our formulas. This advantage seems to be very attractive for practical computations.

Theorem 1 will be essentially used in the next Section, where the convergence criteria of the methods (4) and (5) will be proved. Here we give one more application of formula (24) concerning the problem of positivity of the Fourier means.

If all polynomials  $W_n^\varphi(x)$ ,  $n \in \mathbb{N}$ , are non-negative, they generate the Fourier means that are usually called positive linear operators or positive summation methods. As is known, they find a lot of applications in Approximation Theory and in Numerical Analysis and can be successfully used for the computational needs. The beginning of the intensive study of the positive summation methods is due to P. Korovkin [6], who obtained some fundamental results. For other references and remarks we refer to [4]. The following statement gives a simple criterion of positivity of the methods (4), which are generated by the kernels of type (1).

**Theorem 2.** *Let a function  $\varphi(\xi) \in \mathcal{K}$ . Then  $W_n(\varphi)(x) \geq 0$  for all  $x \in \mathbb{R}^d$  and  $n \in \mathbb{N}$  if and only if  $\widehat{\varphi}(x) \in L_1(\mathbb{R}^d)$  and  $\widehat{\varphi}(x) \geq 0$  in  $\mathbb{R}^d$ . Moreover, in this case  $\|W_n(\varphi)\|_1 = \|\widehat{\varphi}\|_1 = (2\pi)^d$  for each  $n \in \mathbb{N}$ .*

**Proof.** First we proof the necessity. Integrating (1) we have

$$\|W_n(\varphi)\|_1 = \sum_{k \in \mathbb{Z}^d} \varphi\left(\frac{k}{n}\right) \cdot \int_{\mathbb{T}^d} e^{ikx} dx = (2\pi)^d \cdot \varphi(0) = (2\pi)^d, \quad n \in \mathbb{N}.$$

By Theorem 1 this implies that  $\widehat{\varphi} \in L_1(\mathbb{R}^d)$  and

$$1 = (\widehat{\varphi})^\vee(0) = (2\pi)^{-d} \int_{\mathbb{R}^d} \widehat{\varphi}(x) dx \leq (2\pi)^{-d} \cdot \|\widehat{\varphi}\|_{L_1(\mathbb{R}^d)} = 1,$$

that is,

$$\int_{\mathbb{R}^d} (|\widehat{\varphi}(x)| - \widehat{\varphi}(x)) dx = 0$$

and  $\widehat{\varphi}(x) = |\widehat{\varphi}(x)| \geq 0$  in  $\mathbb{R}^d$ . To prove the sufficiency we use Theorem 1 again. We have

$$(2\pi)^d = \int_{\mathbb{T}^d} W_n(\varphi)(x) dx \leq \|W_n(\varphi)\|_1 \leq \|\widehat{\varphi}\|_{L_1(\mathbb{R}^d)} = \int_{\mathbb{R}^d} \widehat{\varphi}(x) dx = (2\pi)^d \cdot (\widehat{\varphi})^\vee(0) = (2\pi)^d$$

and, therefore

$$\int_{\mathbb{T}^d} (|W_n(\varphi)(x)| - W_n(\varphi)(x)) dx = 0$$

that implies  $W_n(\varphi)(x) \geq 0$ . ■

#### 4. THE CONVERGENCE CRITERIA AND THE COMPARISON THEOREM

First we study the Fourier means.

**Theorem 3.** *Let  $\varphi \in \mathcal{K}$ . Then the following statements are equivalent:*

- 1)  $\{\mathcal{F}_n^\varphi\}$  has the convergence property in  $C$  ;
- 2)  $\{\mathcal{F}_n^\varphi\}$  has the convergence property in  $L_1$  ;
- 3)  $\{\mathcal{F}_n^\varphi\}$  has the convergence property in  $L_p$  for all  $1 \leq p \leq +\infty$ ;
- 4)  $\widehat{\varphi}(x) \in L_1(\mathbb{R}^d)$ .

**Proof.** In view of Lemma 1 it is enough to prove that the first and the fourth statement are equivalent.

*Necessity.* If  $\{\mathcal{F}_n^\varphi\}$  has the convergence property, the sequence  $\{\|W_n(\varphi)\|_1\}$  is bounded by virtue of Lemma 1. Then  $\widehat{\varphi}$  belongs to  $L_1(\mathbb{R}^d)$  by Theorem 1.

*Sufficiency.* If  $\widehat{\varphi} \in L_1(\mathbb{R}^d)$ , the sequence  $\{\|W_n(\varphi)\|_1\}$  is bounded by Theorem 1. Moreover, by Lemma 6 the function  $\varphi$  is continuous, in particular, for each  $m \in \mathbb{Z}^d$

$$\lim_{n \rightarrow +\infty} \varphi\left(\frac{m}{n}\right) = \varphi(0) = 1 .$$

Applying Lemma 1 we obtain that  $\{\mathcal{F}_n^\varphi\}$  has the convergence property in  $C$ . ■

As it follows from Theorem 3 the condition " $\widehat{\varphi} \in L_1(\mathbb{R}^d)$ " is sufficient for the convergence of  $\{\mathcal{F}_n^\varphi\}$  in the spaces  $L_p$  with  $1 < p < +\infty$ . The example of the Fourier partial sums that corresponds to the characteristic function of the cube  $[-1, 1]^d$  as  $\varphi$  shows in view of the Riesz theorem (see, for instance, [12, Chapter 7, § 4]) that this condition is not necessary. However, one can point out a simple necessary condition in terms of the Fourier transform of the function  $\varphi$  that turns out to be sharp in a certain sense.

We put

$$\widetilde{p} = \begin{cases} p & , \quad 1 < p \leq 2 \\ \frac{p}{p-1} & , \quad 2 < p < +\infty \end{cases} .$$

**Theorem 4.** *Let  $\varphi \in \mathcal{K}$  and  $1 < p < +\infty$ . If  $\{\mathcal{F}_n^\varphi\}$  has the convergence property in  $L_p$ , then  $\widehat{\varphi}(x) \in L_{\widetilde{p}}(\mathbb{R}^d)$ .*

**Proof.** Let first  $1 < p \leq 2$ . We consider a function  $\psi(\xi) \in \mathcal{K}$ , which is, in addition, infinitely differentiable on  $\mathbb{R}^d$  and  $\psi(\xi) = 1$  for all  $\xi \in \text{supp}\varphi$ .

If  $\{\mathcal{F}_n^\varphi\}$  has the convergence property in  $L_p$ , then by the Banach-Steinhaus theorem the sequence of the norms  $\{\|\mathcal{F}_n^\varphi\|_{(p)}\}$  is bounded and, in particular,

$$\|\mathcal{F}_n^\varphi(W_n(\psi))\|_p \leq c \cdot \|W_n(\psi)\|_p. \quad (25)$$

Here and throughout the proof we denote by  $c$  constants, which are independent of  $n$ . In different estimates they can also be different. Taking into account that  $\widehat{\psi} \in L_1(\mathbb{R}^d)$  we obtain by means of the Nikolsky inequality [9, p. 145] and Theorem 1

$$\|W_n(\psi)\|_p \leq c \cdot n^{d(1-1/p)} \cdot \|W_n(\psi)\|_1 \leq c' \cdot n^{d(1-1/p)}. \quad (26)$$

Since  $\varphi(\xi)\psi(\xi) = \varphi(\xi)$  for all  $\xi \in \mathbb{R}^d$ , we have by virtue of (9)

$$\mathcal{F}_n^\varphi(W_n(\psi); x) = W_n(\varphi)(x), \quad (x \in \mathbb{R}^d). \quad (27)$$

Combining (25)-(27) we get

$$n^{d(1/p-1)} \cdot \|W_n(\varphi)\|_p \leq c,$$

that means that the conditions of Lemma 5 are satisfied; therefore,  $\widehat{\varphi} \in L_p(\mathbb{R}^d)$ .

To treat the case  $2 < p < +\infty$  we apply the standard arguments of duality. In view of what was proved above it is enough to check that

$$\|\mathcal{F}_n^\varphi(W_n(\psi))\|_{\widetilde{p}} \leq c \cdot \|W_n(\psi)\|_{\widetilde{p}}. \quad (28)$$

Using formula (9), the boundedness of the sequence of the norms  $\{\|\mathcal{F}_n^\varphi\|_{(p)}\}$ , Plancherel's equality and Hölder's inequality, we get

$$\begin{aligned} \|\mathcal{F}_n^\varphi(W_n(\psi))\|_{\widetilde{p}} &= \sup_{\|g\|_p \leq 1} |(\mathcal{F}_n^\varphi(W_n(\psi)), g)| = \sup_{\|g\|_p \leq 1} \left| \sum_{k \in \mathbb{Z}^d} \varphi\left(\frac{k}{n}\right) \psi\left(\frac{k}{n}\right) \overline{g^\wedge(k)} \right| = \\ &= \sup_{\|g\|_p \leq 1} |(W_n(\psi), \mathcal{F}_n^\varphi(g))| \leq \|W_n(\psi)\|_{\widetilde{p}} \cdot \|\mathcal{F}_n^\varphi\|_{(p)} \leq c \cdot \|W_n(\psi)\|_{\widetilde{p}}, \end{aligned}$$

where  $g^\wedge(k)$ ,  $k \in \mathbb{Z}^d$ , are the Fourier coefficients of the function  $g$ .

The proof is complete. ■

When considering the Bochner-Riesz means (see below) one can easily prove that the condition  $\widehat{\varphi} \in L_{\widetilde{p}}(\mathbb{R}^d)$  is not sufficient for the convergence in  $L_p$ . On the other hand, we will see that it can not be improved, that is, the number  $\widetilde{p}$  is sharp.

Now we deal with the interpolation means. The criterion of their convergence turns out to coincide with the criterion for the Fourier means.

**Theorem 5.** *Let  $\varphi(\xi) \in \mathcal{K}$ . Then the sequence of the interpolation means  $\{\mathcal{I}_n^\varphi\}$  given by (5) has the convergence property in  $C$  if and only if  $\widehat{\varphi}(x) \in L_1(\mathbb{R}^d)$ .*

**Proof.** The proof of this theorem is quite similar to the proof of Theorem 3. The only difference consists in using Lemma 2 instead of Lemma 1. ■

Now we give some applications. Noticing that the Fourier transforms of functions generating the classical kernels can be calculated exactly, the conclusions on their convergence can be done immediately.

For this purpose we set down the following table.

<i>Kernel</i>	<i>d</i>	$\varphi(\xi)$	$\widehat{\varphi}(x)$	$p : \widehat{\varphi}(x) \in L_p(\mathbb{R}^d)$
Dirichlet	1	1	$\frac{2 \sin x}{x}$	$1 < p \leq +\infty$
Fejer	1	$1 -  \xi $	$\frac{4 \sin^2 \frac{x}{2}}{x^2}$	$\frac{1}{2} < p \leq +\infty$
Rogosinski	1	$\cos \frac{\pi \xi}{2}$	$\frac{\pi \cos x}{((\frac{\pi}{2})^2 - x^2)}$	$\frac{1}{2} < p \leq +\infty$
Vallee-Poussin	1	$\begin{cases} 1, &  \xi  \leq 1 \\ 2 -  \xi , & 1 <  \xi  \leq 2 \end{cases}$	$\frac{4 \sin \frac{x}{2} \sin \frac{3x}{2}}{x^2}$	$\frac{1}{2} < p \leq +\infty$
Bochner-Riesz	any	$(1 -  \xi ^2)^\delta, (\delta \geq 0)$	$c \frac{J_{\delta+d/2}( x )}{ \xi ^{\delta+d/2}}$	$\frac{2d}{d+2\delta+1} < p \leq +\infty$

Table 1.

The first column contains the names of the kernels. The second column is intended for the dimension. The third and the fourth entries are generators and their Fourier transforms respectively and the fifth column contains the ranges of  $p$ , for which  $\widehat{\varphi} \in L_p(\mathbb{R}^d)$ . The values of the function  $\varphi(\xi)$  are normally given in  $D_1 = \{\xi : |\xi| \leq 1\}$ . Outside we put  $\varphi(\xi) = 0$ . For the Vallee-Poussin kernels  $\varphi(\xi)$  is equal to 0 outside  $[-2, 2]$ .

It should be noticed that practically all kernels which we call classical are one-dimensional. We do not take into account here their tensor products that can be considered as a trivial generalization to the multivariate case. One of the most remarkable exceptions are the Bochner-Riesz means, whose properties

essentially depend on the dimension.

In the last line  $J_s(x)$ ,  $s > -1/2$ , is the Bessel function,  $c \equiv c(\delta) = \pi^{-\delta} \cdot \Gamma(\delta+1)$ . The Fourier transform of Bochner-Riesz kernel is given in [11, Chapter 9, § 2.2, pp. 389-390]. The range of  $p$  for which it belong to  $L_p(\mathbb{R})$  can be immediately obtained from the asymptotic formula for the Bessel functions (see, for instance, [11, Chapter 8, § 5, pp. 356-357]). It is clear that the Fourier transform of the Bochner-Riesz kernel belongs to  $L_1(\mathbb{R}^d)$  if and only if

$$\delta > \frac{d-1}{2}. \quad (29)$$

The number  $(d-1)/2$  is well-known as the critical index for the convergence of the Bochner-Riesz means. Initially it was defined as the infimum of all  $\delta$ , for which the condition (17) is satisfied with some  $\varepsilon > 0$  ([12, Chapter 7, § 2, p. 255]). By Theorems 3 and 5 we immediately obtain another proof of the classical result that the Bochner-Riesz means converge in  $C, L_1$  or in  $L_p$  for all  $1 \leq p \leq +\infty$  if and only if the condition (29) is satisfied (see, for instance, [12, Chapter 7, §§ 2-4]).

The number  $1/2$  as a lower bound for  $p$  does naturally arise when the Fejer or the Vallee-Poissin kernels are applied in problems of representation of functions from  $L_p$  with  $0 < p < 1$  by trigonometric series (see, for instance, [8]) or by the Fourier means if a given function satisfies a priori some additional conditions [13]. In forthcoming papers we will show that the numbers arising in such way have a deeper sense in Approximation Theory. More precisely, they turn out to be sharp lower bounds for that  $p$ , for which the corresponding families of linear polynomial operators have the convergence property in  $L_p$  (see, for instance, [1]).

Now we return to Theorem 4. Applying it to the Bochner-Riesz means with  $\delta = 0$  and  $d \geq 2$  we obtain that they do not have the convergence property in  $L_p$  with  $p$  outside the interval  $(2d/(d+1), 2d/(d-1))$ . The non-periodic counterpart of this fact is due to C. Herz. However, as it was proved by G. Fefferman, the convergence property in this case holds only for  $p = 2$  (for references and more details we refer to [11, Chapter 9]). Thus, the condition in Theorem 4, being necessary, is not sufficient for the convergence in  $L_p$  with  $1 < p < +\infty$ . On the other hand, if, for example,  $d = 2$ , then the Bochner-Riesz means have the convergence property in the range  $4/(3+2\delta) < p < 4/(1-2\delta)$ , whenever  $0 < \delta \leq 1/2$  [11, Chapter 9, pp. 389-390]. This is also a consequence of Theorem 4 for  $d = 2$  and  $0 < \delta \leq 1/2$ . This observation shows that the index  $\tilde{p}$  is sharp in Theorem 4. Similar statements turn out to be valid for  $d \geq 3$  as well. Indeed, if  $(d-1)/2(d+1) < \delta < (d-1)/2$ , the convergence property holds for  $2d/(d+2\delta+1) < p < 2d/(d-1-2\delta)$  [11, Chapter 9, pp. 390-391, Figure 3]. The range of  $p$ , for which  $\widehat{\varphi} \in L_{\tilde{p}}(\mathbb{R}^d)$  is the same (see Table 1).

We emphasize that everything what was said about the Fourier means does automatically hold for their interpolation counterparts because of Theorem 5 which shows that the convergence criterions of both methods are similar. The



following result strengthens this observation. More precisely, it turns out that even the approximation errors of the methods are equivalent to each other in case of convergence.

**Theorem 6.** *Let  $\varphi(\xi) \in \mathcal{K}$  and  $\widehat{\varphi}(x) \in L_1(\mathbb{R}^d)$ . Then*

$$\|f - \mathcal{I}_n^\varphi(f)\| \asymp \|f - \mathcal{F}_n^\varphi(f)\|, \quad (f \in C, n \in \mathbb{N}). \quad (30)$$

**Proof.** Since  $\widehat{\varphi}(x) \in L_1(\mathbb{R}^d)$ , the methods  $\{\mathcal{F}_n\}$  and  $\{\mathcal{I}_n\}$  converge in  $C$  by Theorems 3 and 5 and the sequence  $\{\|W_n(\varphi)\|_1\}$  is bounded by virtue of Theorem 1. Applying estimates (7) and (12) in combination with (24) we get

$$\|\mathcal{F}_n\| = (2\pi)^{-d} \cdot \|\widehat{\varphi}\|_{L_1(\mathbb{R}^d)}, \quad n \in \mathbb{N}; \quad (31)$$

$$\|\mathcal{I}_n\| \leq 3^d (2\pi)^{-d} \cdot \|\widehat{\varphi}\|_{L_1(\mathbb{R}^d)}, \quad n \in \mathbb{N}, \quad (32)$$

for the operator norms of the Fourier means and the interpolation means, respectively.

Further we will follow the proof given in [3] in the non-periodic case. Let  $f \in C$ . Since the functions  $\mathcal{F}_n(f; x)$  and  $\mathcal{I}_n(f; x)$  belong to  $\mathcal{T}_N$ , we get by comparing formulas (9) and (15) that

$$\mathcal{F}_n^\varphi \circ \mathcal{I}_n^\varphi(f; x) = \mathcal{I}_n^\varphi \circ \mathcal{I}_n^\varphi(f; x); \quad \mathcal{I}_n^\varphi \circ \mathcal{F}_n^\varphi(f; x) = \mathcal{F}_n^\varphi \circ \mathcal{F}_n^\varphi(f; x), \quad (x \in \mathbb{R}^d). \quad (33)$$

Using (33) we get

$$\begin{aligned} \|f - \mathcal{F}_n^\varphi(f)\| &\leq \|f - \mathcal{I}_n^\varphi(f)\| + \|\mathcal{I}_n^\varphi(f) - \mathcal{I}_n^\varphi \circ \mathcal{I}_n^\varphi(f)\| + \|\mathcal{F}_n^\varphi \circ \mathcal{I}_n^\varphi(f) - \mathcal{F}_n^\varphi(f)\| \leq \\ &\leq \{1 + \|\mathcal{I}_n^\varphi\| + \|\mathcal{F}_n^\varphi\|\} \cdot \|f - \mathcal{I}_n^\varphi(f)\|. \end{aligned}$$

Changing the roles of  $\mathcal{F}_n^\varphi$  and  $\mathcal{I}_n^\varphi$  we obtain the second part of equivalence (30).

Using (31) and (32) we can give some estimates for the constants. More exactly,

$$c(d; \varphi)^{-1} \cdot \|f - \mathcal{F}_n^\varphi(f)\| \leq \|f - \mathcal{I}_n^\varphi(f)\| \leq c(d; \varphi) \cdot \|f - \mathcal{F}_n^\varphi(f)\|, \quad (f \in C, n \in \mathbb{N}),$$

where

$$c(d; \varphi) = 1 + (2\pi)^{-d} (3^d + 1) \cdot \|\widehat{\varphi}\|_{L_1(\mathbb{R}^d)}.$$

The proof is complete. ■

Now we give some final remarks.

The number  $N$  given by (2) and (3) depends on  $\sigma \geq \sigma_\varphi$ . However,  $\sigma$  does not influence the equivalence and does not affect the constants. This fact becomes clear, if we notice that the interpolation means given by (5) tend to the corresponding Fourier means, if  $\sigma$  (and therefore  $N$ ) tends to  $+\infty$ .

The condition " $\widehat{\varphi} \in L_1(\mathbb{R}^d)$ " is essential in Theorem 6. Indeed, if  $\varphi(\xi) = \mathcal{X}_{[-1,1]}(\xi)$ , that is, the characteristic function of the interval  $[-1, 1]$ , the convergence properties of the Fourier partial sums and the Lagrange polynomials of trigonometric interpolation, which correspond to the methods (4) and (5) in this case, can be different (see, for instance [15, Vol. 2, Chapter X, §5, pp 16]), in particular, one can not expect that the equivalence (30) holds.

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# Admissibility of time-varying observations for time-varying systems

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## Abstract

By using the theory of evolution semigroups we characterize the admissibility of time-varying observations for time-varying systems by that of some time-invariant Weiss's observations for time-invariant systems (cf. [22]). This characterization gives an opportunity to apply known results for time-invariant systems to the time-varying situation. An example of time-varying heat conduction process is given to illustrate our framework.

**Keywords and phrases:** Admissibility of time-varying observations, evolution semigroups, output functions.

**AMS subject classification (1991):** 93B17, 93C25, 93C50, 47D06, 35K10.

## 1. INTRODUCTION

In this paper we consider time-varying infinite dimensional systems of the form

$$\begin{aligned} \dot{x}(t) &= A(t)x(t), \quad x(s) = x \quad \text{for } t \geq s, \\ y(t) &= C(t)x(t). \end{aligned} \tag{1.1}$$

Here,  $A(t) : X \rightarrow X$  are unbounded linear operators and  $C(t) \in \mathcal{L}(W, Y)$ , where  $W$  is a subspace of the Banach space  $X$  and  $Y$  is a Banach space. The space  $X$  (resp.  $Y$ ) is called the *state space* (resp. the *output space*),  $C(\cdot)$  is the *observation family* of the system (1.1) and  $y(\cdot)$  is the *output function* associated to  $s$  and  $x$ .

It is well-known that the solution of (1.1) (if it exists) is obtained by an evolution family  $(\phi(t, s))_{t \geq s}$ . Hence the output function is given by

$$y(t) = C(t)\phi(t, s)x. \tag{1.2}$$

This function is only obtained when the state of the system is in the domain  $W$ , i.e., the observation is not possible for all initial states  $x$ . So our aim in this work is to extend the observation to all of the state space  $X$  and to give another interpretation of (1.2) even if the state is not in  $W$ . On any finite time interval the output function should be an  $L^p$ -function for some  $p \in [1, \infty]$  and should depend continuously on the initial state  $x$ . For that we need the observation family to be *admissible*, see Definition 3.1.

By using the evolution semigroup associated to the evolution family  $\phi(\cdot, \cdot)$  defined in a larger Banach function space, we give a characterization of admissibility, which we have adapted for time-varying observations of time-varying systems from the theory of time-invariant Weiss systems. This characterization together with some results established by Weiss, allow to give another interpretation of the output function  $y(\cdot)$ , see Theorem 4.3.

The problem of admissibility has been studied by many authors, e.g., [3], [22], [5], and [15], but they consider time-invariant systems with time-invariant observations. There are

some works that consider admissibility, in some sense, for time-varying systems, as an assumption to study robust stability [12], [14] or the linear quadratic control problem [4].

The unboundedness of the operators  $C(t)$  considered here arises naturally when we deal with distributed systems, see Examples 2.2 and 2.3, when observations are limited to regions of the boundary or manifolds of lower dimension interior to the domain of the PDE, or point observations for systems described by linear PDE's.

Our paper is organized as follows. In §2 we give, as a preliminary, the definition of the admissibility of time-invariant observations for time-invariant systems with some examples. In §3 we introduce the definition of admissibility for the observation family associated to an evolution family  $\phi(\cdot, \cdot)$ . The transformation of the problem to some time-invariant Weiss systems is given. In §4 we use the transformation obtained in §2 to give another interpretation of output functions via some of Weiss's results and finally, an example of time-varying heat conduction process is given in §5 to illustrate our framework.

## 2. PRELIMINARIES

This section is meant to recall the notion of admissibility of observation operators associated to time-invariant systems. This notion was introduced by Pritchard and Salamon [17], and used by many authors, e.g., [8], [22] and [11], to study the well posedness, exact observability, stabilization and quadratic control problems, for time-invariant systems with unbounded observation operators.

Let  $(W, \|\cdot\|_W)$ ,  $(X, \|\cdot\|)$  and  $(Y, \|\cdot\|_Y)$  be Banach spaces such that

$$W \xhookrightarrow{d} X.$$

This means that the embedding is continuous and  $W$  is dense in  $X$ .

Let  $S(\cdot) := (S(t))_{t \geq 0}$  be a strongly continuous semigroup on  $X$ . Suppose that  $W$  is invariant under the semigroup  $(S(t))_{t \geq 0}$ , i.e.,  $S(t)W \subset W$  for all  $t \geq 0$ .

We begin by giving the definition, introduced by Weiss [22], of the admissibility of time-invariant observation operators with respect to time-invariant systems.

**Definition 2.1.** *Let  $C$  be a bounded linear operator from  $W$  into  $Y$ . We say that  $C$  is an admissible observation operator (a.o.o.) for  $(W, X, Y, p, S(\cdot))$ ,  $p \in [1, \infty]$ , if for some  $T_0 > 0$  there is an  $\alpha_0 > 0$  such that*

$$\|CS(\cdot)x\|_{L^p(0, T_0; Y)} \leq \alpha_0 \|x\| \quad \text{for all } x \in W. \quad (2.1)$$

The following simple examples show us the dependence of the admissibility of the choice of  $p \in [1, \infty]$ .

**Example 2.2.** *We consider the heat conduction system with Dirichlet boundary conditions*

$$\dot{x}(t) = Ax(t), \quad t \geq 0, \quad x(0) = x_0,$$

where  $A$  is given by

$$D(A) := H^2(0, 1) \cap H_0^1(0, 1) \quad \text{and} \quad Af := f'' \quad \text{for } f \in D(A).$$

Let  $S(\cdot)$  be the analytic semigroup generated by  $A$  on  $X := L^2(0, 1)$ ,  $\xi_0 \in (0, 1)$  and consider the observation operator  $Cf := f(\xi_0)$ , which gives the value of the temperature of the system at  $\xi_0$ . This operator is clearly an unbounded operator on  $L^2(0, 1)$ .

Curtain and Pritchard (see [3], p. 216) showed that, if  $p < 4$ , then  $C$  is admissible for  $(C[0, 1], L^2(0, 1), \mathbb{R}, p, S(\cdot))$ .

**Example 2.3.** Let  $a_{ij}(\cdot)$  be real-valued functions satisfying  $a_{ij}(\cdot) = a_{ji}(\cdot) \in W^{1,\infty}(\mathbb{R}^n)$  for  $i, j = 1, \dots, n$  and suppose that there is a constant  $c > 0$  such that

$$\sum_{i,j=1}^n a_{ij}(x) y_i y_j \geq c|y|^2 \quad \text{for all } x, y \in \mathbb{R}^n. \quad (2.2)$$

On the Hilbert space  $L^2(\mathbb{R}^n)$  we consider the operators  $A$  and  $C$  given by

$$\begin{aligned} Au &:= \sum_{i,j=1}^n D_i(a_{ij}(\cdot) D_j u) \text{ with } D(A) := H^2(\mathbb{R}^n) \text{ and} \\ Cu &:= \sum_{i=1}^n c_i(\cdot) D_i u \text{ with } D(C) := H^1(\mathbb{R}^n), \end{aligned}$$

where  $c_i \in L^\infty(\mathbb{R}^n)$  and  $D_i := \frac{\partial}{\partial x_i}$  for  $i = 1, \dots, n$ . Then  $C \in \mathcal{L}(H^1(\mathbb{R}^n), L^2(\mathbb{R}^n))$  and it is well-known that  $A$  generates a bounded analytic semigroup  $S(\cdot)$  on  $L^2(\mathbb{R}^n)$  (cf. [9], Proposition VI-5.22). From (2.2) and the analyticity of  $S(\cdot)$  we obtain

$$\begin{aligned} \|D_i S(t)u\|_{L^2}^2 &= \int_{\mathbb{R}^n} |D_i S(t)u(x)|^2 dx \\ &\leq \int_{\mathbb{R}^n} \sum_{i=1}^n |D_i S(t)u(x)|^2 dx \\ &\leq \frac{1}{c} \int_{\mathbb{R}^n} \sum_{i,j=1}^n a_{ij}(x) D_i S(t)u(x) D_j S(t)u(x) dx \\ &= \frac{1}{c} (-AS(t)u | S(t)u)_{L^2} \leq \frac{M}{ct} \|S(t)u\|_{L^2} \|u\|_{L^2} \\ &\leq \frac{\tilde{M}}{ct} \|u\|_{L^2}^2 \end{aligned}$$

for  $t > 0$ ,  $u \in L^2(\mathbb{R}^n)$  and some constant  $\tilde{M} > 0$ . So, since  $c_i \in L^\infty(\mathbb{R}^n)$ , there exists a constant  $K > 0$  such that

$$\|CS(t)u\|_{L^2} \leq \frac{K}{\sqrt{t}} \|u\|_{L^2}.$$

Thus,  $C$  is admissible for  $(H^1(\mathbb{R}^n), L^2(\mathbb{R}^n), L^2(\mathbb{R}^n), p, S(\cdot))$  if  $p < 2$ .

Now we give another equivalent definition of admissibility which is more precise.

**Proposition 2.4.** The operator  $C \in \mathcal{L}(W, Y)$  is an a.o.o. for  $(W, X, Y, p, S(\cdot))$  if and only if there is an increasing function  $\delta : (0, \infty) \rightarrow \mathbb{R}_+$  such that

$$\|CS(\cdot)x\|_{L^p(0,T;Y)} \leq \delta(T) \|x\| \quad \text{for all } x \in W \text{ and } T > 0.$$

*Proof.* We only have to show that the condition is necessary. First, it is clear that the inequality (2.1) holds for  $\alpha_0$  and all  $0 < T \leq T_0$ . Let us consider now  $T > T_0$  and  $n \in \mathbb{N}$  such that  $T \in [nT_0, (n+1)T_0)$ . Then we obtain

$$\begin{aligned} \|CS(\cdot)x\|_{L^p(0,T;Y)}^p &= \sum_{k=0}^{n-1} \|CS(\cdot)x\|_{L^p(kT_0,(k+1)T_0;Y)}^p + \|CS(\cdot)x\|_{L^p(nT_0,T;Y)}^p \\ &= \sum_{k=0}^{n-1} \|CS(\cdot + kT_0)x\|_{L^p(0,T_0;Y)}^p + \|CS(\cdot + nT_0)x\|_{L^p(0,T-nT_0;Y)}^p \\ &\leq \left(\sum_{k=0}^n \|S(kT_0)x\|^p\right) \alpha_0^p \\ &\leq \left(\sum_{k=0}^n \|S(kT_0)\|^p\right) \alpha_0^p \|x\|^p. \end{aligned} \quad (2.3)$$

For  $p = \infty$ , we proceed in the same manner and we obtain

$$\|CS(\cdot)x\|_{L^\infty(0,T;Y)} \leq \alpha_0 \sum_{k=0}^n \|S(kT_0)\| \|x\|. \quad (2.4)$$

Thus (2.1) holds for all  $T > 0$  and  $\alpha_0$  replaced by a constant  $\alpha_0(T)$ . Hence the operator  $R_T$  defined by

$$(R_T x)(\cdot) := CS(\cdot)x \text{ for } x \in W$$

is bounded from  $(W, \|\cdot\|)$  to  $L^p(0, T; Y)$  for each  $T > 0$ .

If we put

$$\delta(T) := \|R_T\|_{\mathcal{L}((W, \|\cdot\|), L^p(0, T; Y))} \text{ for } T > 0,$$

we can easily verify that  $\delta$  is the function which we are looking for.  $\square$

The function  $\delta$  defined above is exponentially bounded as shows the following remark.

**Remark 2.5.** Let  $M \geq 1$  and  $\omega > 0$  such that  $\|S(t)\|_{\mathcal{L}(X)} \leq Me^{\omega t}$  for all  $t \geq 0$ . Then by (2.3) and (2.4) we have

$$\delta(T) \leq M_1 e^{\omega T} \text{ for all } T > 0,$$

where

$$M_1 := \begin{cases} \left(\frac{1}{e^{p\omega T_0} - 1}\right)^{\frac{1}{p}} M e^{\omega T_0} \alpha_0, & \text{if } p \in [1, \infty) \\ \frac{M e^{\omega T_0} \alpha_0}{e^{\omega T_0} - 1}, & \text{if } p = \infty. \end{cases}$$

### 3. ADMISSIBILITY OF TIME-VARYING OBSERVATIONS FOR TIME-VARYING SYSTEMS

In this section we introduce the admissibility of observations of time-varying systems, and we discuss the relationships between this notion and that given in §2. First of all, we introduce the family  $(\phi(t, s))_{(t,s) \in \Delta_J}$  of linear and bounded operators on  $X$  satisfying the following properties

- (i)  $\phi(t, t) = I$ ,  $\phi(t, \tau) \phi(\tau, s) = \phi(t, s)$  for  $(t, \tau), (\tau, s) \in \Delta_J$ ,
- (ii)  $\Delta_J \ni (t, s) \mapsto \phi(t, s)x$  is continuous for all  $x \in X$ ,

where  $J$  is an interval in  $\mathbb{R}_+$  and  $\Delta_J := \{(t, s) \in J^2 : t \geq s\}$ . Such a family is called *evolution family*. We say that  $(\phi(t, s))_{(t,s) \in \Delta_J}$  is *exponentially bounded* if there is  $M \geq 1$ ,  $\omega \in \mathbb{R}$  such that  $\|\phi(t, s)\| \leq M e^{\omega(t-s)}$  for all  $(t, s) \in \Delta_J$ .



**Definition 3.1.** Let consider the same spaces as in Section 2 and  $\mathcal{C} := (C(t))_{t \geq 0}$  be in the space  $C(\mathbb{R}_+, \mathcal{L}_s(W, Y))$  of strongly continuous  $\mathcal{L}(W, Y)$ -valued operator functions on  $\mathbb{R}_+$ . Let  $\phi := (\phi(t, s))_{(t,s) \in \Delta_{\mathbb{R}_+}}$  be an evolution family on  $X$  which leaves the space  $W$  invariant. The family  $\mathcal{C}$  is called an admissible observation family (a.o.f.) for  $(W, X, Y, p, \phi)$  if there exists an increasing function  $\gamma : (0, \infty) \rightarrow \mathbb{R}_+$  such that

$$\|C(\cdot)\phi(\cdot, s)x\|_{L^p(s, T; Y)} \leq \gamma(T - s) \|x\| \quad \text{for all } x \in W \text{ and } (T, s) \in \Delta_{\mathbb{R}_+}. \quad (3.1)$$

**Remark 3.2.** a. If  $C$  and  $S(\cdot)$  satisfy (2.1), then the time-invariant family  $\mathcal{C} \equiv C$  is an a.o.f. for  $(W, X, Y, p, \phi)$ , where  $\phi(t, s) := S(t - s)$ .  
b. The notion of admissibility for an observation family has also been studied in Jacob [13], Schnaubelt [20] and has been used as an assumption in [12] and [14].

We now try to establish the relationship between the above notion of admissibility of observations of time-varying systems and the one of time-invariant systems given in §2. For that we will use the theory of evolution semigroups.

For any Banach space  $F$  and any interval  $J = [a, b] \subset \mathbb{R}_+$ ,  $b \in \mathbb{R}_+ \cup \{\infty\}$ , we define the Banach spaces

$$F(J, p) := \begin{cases} L^p(J, F) & \text{if } p \in [1, \infty) \\ \left\{ \begin{array}{ll} \{f \in C(J, F); f(a) = 0\} & \text{if } b < \infty \\ \{f \in C(J, F); f(a) = \lim_{s \rightarrow \infty} f(s) = 0\} & \text{if } b = \infty \end{array} \right\} & \text{if } p = \infty \end{cases}$$

endowed with the  $L^p$ -norm  $\|\cdot\|_p$ , if  $1 \leq p < \infty$ , and the sup-norm  $\|\cdot\|_\infty$ , if  $p = \infty$ , respectively.

Let  $(\phi(t, s))_{(t,s) \in \Delta_J}$  be an evolution family on  $X$ . We define for each  $t \geq 0$  the operator  $E_J(t)$  on  $X(J, p)$  by

$$(E_J(t)f)(s) := \begin{cases} \phi(s, s-t)f(s-t), & s-t, s \in J \\ 0, & s \in J, s-t \notin J. \end{cases}$$

The family  $(E_J(t))_{t \geq 0}$  defines a semigroup on  $X(J, p)$  called *evolution semigroup* associated to the evolution family  $(\phi(t, s))_{(t,s) \in \Delta_J}$ .

**Proposition 3.3.** If the evolution family  $(\phi(t, s))_{(t,s) \in \Delta_J}$  is exponentially bounded, then  $E_J := (E_J(t))_{t \geq 0}$  is a strongly continuous semigroup on  $X(J, p)$  for all  $p \in [1, \infty]$ .

The proof of this result is given in [19], Proposition 1.9, and for more information on evolution semigroups we refer to, e.g. [18], [2] and the references therein.

**Remark 3.4.** If  $J$  is a compact interval, then by the principle of uniform boundedness the evolution family  $(\phi(t, s))_{(t,s) \in \Delta_J}$  is bounded. So, by Proposition 3.3,  $(E_J(t))_{t \geq 0}$  is a strongly continuous semigroup on  $X(J, p)$  for all  $p \in [1, \infty]$ .

We can now state the main result of this section, which claims that the admissibility for time-varying systems is equivalent to the admissibility of a certain observation operator for time-invariant systems.

**Theorem 3.5.** *Let  $X, W, Y$  be as above and  $\phi := (\phi(t, s))_{(t,s) \in \Delta_{\mathbb{R}_+}}$  be an evolution family on  $X$  which leaves  $W$  invariant. Let  $\mathcal{C} \in C(\mathbb{R}_+, \mathcal{L}_s(W, Y))$ . If the restriction  $\phi|_W$  forms an evolution family on  $W$ , then the following assertions are equivalent.*

- (i)  $\mathcal{C}$  is an a.o.f. for  $(W, X, Y, p, \phi)$ .
- (ii) For each compact interval  $J$  in  $\mathbb{R}_+$ , the bounded multiplication operator  $\Gamma_J$  defined from  $W(J, p)$  into  $Y(J, p)$  by

$$(\Gamma_J f)(s) := C(s) f(s) \text{ for } s \in J \text{ and } f \in W(J, p)$$

is a.o.o. for  $(W(J, p), X(J, p), Y(J, p), p, E_J(\cdot))$ .

*Proof.* 1<sup>st</sup> case:  $p \in [1, \infty)$ .

(i)  $\Rightarrow$  (ii) Let  $x \in W$  and  $J := [a, b]$ ,  $\tilde{J}$  be intervals on  $\mathbb{R}_+$  such that  $\tilde{J} \subset J$ . By setting  $f := \chi_{\tilde{J}}(\cdot)x \in W(J, p)$ , where  $\chi_{\tilde{J}}(\cdot)$  is the characteristic function associated to  $\tilde{J}$ , we obtain

$$\begin{aligned} \|\Gamma_J E_J(\cdot) f\|_{L^p(0, T; Y(J, p))}^p &= \int_0^T \int_J \|C(s) \chi_J(s-t) \phi(s, s-t) f(s-t)\|_Y^p ds dt \\ &= \int_0^T \int_{a-t}^{b-t} \|\chi_J(s) C(s+t) \phi(s+t, s) f(s)\|_Y^p ds dt \\ &\leq \int_0^T \int_J \|C(s+t) \phi(s+t, s) f(s)\|_Y^p ds dt \\ &= \int_0^T \int_{\tilde{J}} \|C(s+t) \phi(s+t, s) x\|_Y^p ds dt. \end{aligned}$$

By Fubini's theorem,

$$\|\Gamma_J E_J(\cdot) f\|_{L^p(0, T; Y(J, p))}^p \leq \int_{\tilde{J}} \int_0^T \|C(s+t) \phi(s+t, s) x\|_Y^p dt ds.$$

Hence, by (i), we obtain

$$\|\Gamma_J E_J(\cdot) f\|_{L^p(0, T; Y(J, p))}^p \leq \gamma(T)^p \|f\|_{X(J, p)}^p.$$

This inequality holds also for all step functions in  $W(J, p)$ . Therefore (ii) is obtained by the density of  $W$ -valued step functions in both  $W(J, p)$  and  $X(J, p)$ .

(ii)  $\Rightarrow$  (i) Let  $x \in W$ ,  $s_0 \in \mathbb{R}_+$  and  $T > 0$ . Then

$$\begin{aligned} \|C(\cdot) \phi(\cdot, s_0) x\|_{L^p(s_0, s_0+T; Y)}^p &= \int_{s_0}^{s_0+T} \|C(t) \phi(t, s_0) x\|_Y^p dt \\ &= \int_0^T \|C(s_0+t) \phi(s_0+t, s_0) x\|_Y^p dt. \end{aligned}$$

On the other hand, let  $J := [a, b]$  be a compact interval in  $\mathbb{R}_+$ ,  $\tilde{J} := [a, b+T]$  and consider the function

$$f(s) := \begin{cases} x & \text{if } s \in J \\ 0 & \text{if } s \in \tilde{J} \setminus J. \end{cases}$$

Since  $f \in W(\tilde{J}, p)$ , we can write

$$\begin{aligned} \|\Gamma_{\tilde{J}} E_{\tilde{J}}(\cdot) f\|_{L^p(0, T; Y(\tilde{J}, p))}^p &= \int_0^T \int_{\tilde{J}} \|\chi_{\tilde{J}}(s-t) C(s) \phi(s, s-t) f(s-t)\|_Y^p ds dt \\ &= \int_0^T \int_a^{b+T-t} \|C(s+t) \phi(s+t, s) f(s)\|_Y^p ds dt \\ &= \int_0^T \int_J \|C(s+t) \phi(s+t, s) x\|_Y^p ds dt. \end{aligned}$$

By Fubini's theorem we obtain

$$\|\Gamma_{\bar{J}} E_{\bar{J}}(\cdot) f\|_{L^p(0,T;Y(\bar{J},p))}^p = \int_J \int_0^T \|C(s+t) \phi(s+t, s) x\|_Y^p dt ds. \quad (3.2)$$

We set

$$h(s) := \int_0^T \|C(s+t) \phi(s+t, s) x\|_Y^p dt \text{ for } s \in J.$$

Since  $\phi$  is also an evolution family in  $W$ , this yields that  $h \in C(J)$ . Hence we can find  $s_J \in J$  such that

$$h(s_J) = \min_{s \in J} h(s).$$

By (3.2) and (ii) it follows that

$$\begin{aligned} h(s_J) &\leq \frac{1}{|J|} \|\Gamma_{\bar{J}} E_{\bar{J}}(\cdot) f\|_{L^p(0,T;Y(\bar{J},p))}^p \\ &\leq \frac{\delta(T)^p}{|J|} \|f\|_{X(\bar{J},p)}^p \\ &= \delta(T)^p \|x\|^p. \end{aligned}$$

Now we turn back to prove (i).

Consider  $J_n := [s_0, s_0 + \frac{1}{n}]$ ,  $n \in \mathbb{N}$ . Then there is  $s_n \in J_n$  such that

$$\int_0^T \|C(s_n+t) \phi(s_n+t, s_n) x\|_Y^p dt \leq \delta(T)^p \|x\|^p.$$

Therefore, by Lebesgue's theorem, we can conclude that

$$\int_0^T \|C(s_0+t) \phi(s_0+t, s_0) x\|_Y^p dt \leq \delta(T)^p \|x\|^p.$$

$2^{nd}$  case:  $p = \infty$ .

To prove (i)  $\Rightarrow$  (ii) it suffices to show that there is an increasing and positive function  $\delta$  such that

$$\sup_{s \in (J+t) \cap J} \|C(s) \phi(s, s-t) f(s-t)\|_Y \leq \delta(T) \|f\|_{X(J,\infty)},$$

for all  $t \in [0, T]$  and  $f \in W(J, \infty)$ . So, let  $t \in [0, T]$ ,  $s \in J$  such that  $s-t \in J$ . Hence, by (i), we can write

$$\begin{aligned} \|C(s) \phi(s, s-t) f(s-t)\|_Y &= \|C(s) \phi(s, s_0) \phi(s_0, s-t) f(s-t)\|_Y \\ &\leq \gamma(s-s_0) \|\phi(s_0, s-t) f(s-t)\| \\ &\leq \gamma(T) N(J) \|f\|_{X(J,\infty)} \end{aligned} \quad (3.3)$$

for some  $s_0 \in [s-t, s]$ , where  $N(J) := \sup_{(t,s) \in \Delta_J} \|\phi(t, s)\|_{\mathcal{L}(X)}$  which is finite by Remark 3.4. So we can take  $\delta(T) := \gamma(T) N(J)$ .

(ii)  $\Rightarrow$  (i) Let  $0 \leq s_0 < s < T$ ,  $x \in W$  and consider  $f \in W(J, \infty)$  with  $f(s) = x$  and  $\|f\|_{X(J, \infty)} = \|x\|$ , where  $J := [s_0, T]$ . According to (ii) we obtain

$$\begin{aligned} \|C(t) \phi(t, s) x\|_Y &= \|C(t) \phi(t, s) f(s)\|_Y \\ &\leq \|\Gamma_J E_J(\cdot) f\|_{L^\infty(0, T-s; Y(J, \infty))} \\ &\leq \delta(T-s) \|f\|_{X(J, \infty)} \\ &= \delta(T-s) \|x\| \end{aligned}$$

for all  $t \in [s, T]$ . This ends the proof of the theorem.  $\square$

**Remark 3.6.** a. It follows from (3.3) that if  $\phi$  is exponentially bounded in  $X$ , then the function  $(\delta(T))_{T>0}$  can be chosen independently of  $J$  for all  $p \in [1, \infty]$ .

b. The assumption  $\mathcal{C} \in C(\mathbb{R}_+, \mathcal{L}_s(W, Y))$  is only used in the case  $p = \infty$ . For  $p \in [1, \infty)$ , we may instead in Theorem 3.5 that  $\mathcal{C} \in L_{loc}^\infty(\mathbb{R}_+, \mathcal{L}_s(W, Y))$ , the space of strongly measurable functions  $\mathcal{F}$  such that

$$\operatorname{ess\,sup}_{\sigma \in [0, t]} \|\mathcal{F}(\sigma)\|_{\mathcal{L}(W, Y)} < \infty \quad \text{for every } t > 0.$$

c. In Theorem 3.5, if in particular  $X, W$  and  $Y$  are Hilbert spaces and  $p = 2$ , then (i) is equivalent to the fact that for each compact interval  $J$  in  $\mathbb{R}_+$  the system

$$\begin{cases} \xi(t) &= E_J(t) \xi_0, \quad \xi_0 \in X(\mathbb{R}_+, 2), \quad t \geq 0 \\ \eta(t) &= \Gamma_J \xi(t) \end{cases}$$

is of Pritchard-Salamon type, with control operator  $B = 0$ , (cf. [17] or [6]).

If we suppose now that the evolution family  $\phi$  is exponentially bounded, then we obtain the following characterization.

**Corollary 3.7.** Let  $\phi$  be an exponentially bounded evolution family in both  $X$  and  $W$  and let  $\mathcal{C}$  be in the space  $C_b(\mathbb{R}_+, \mathcal{L}_s(W, Y))$  of bounded operator functions in  $C(\mathbb{R}_+, \mathcal{L}_s(W, Y))$ . Then the following properties are equivalent.

- (j)  $\mathcal{C}$  is an a.o.f. for  $(W, X, Y, p, \phi)$ .
- (jj) The multiplication operator  $\Gamma_{\mathbb{R}_+}$ , defined as in Theorem 3.5, is an a.o.o. for  $(W(\mathbb{R}_+, p), X(\mathbb{R}_+, p), Y(\mathbb{R}_+, p), p, E_{\mathbb{R}_+}(\cdot))$ .

*Proof.* Proving this corollary is equivalent to proving that (ii)  $\Leftrightarrow$  (jj), where (ii) is the assertion given in Theorem 3.5.

(jj)  $\Rightarrow$  (ii) For  $p \in [1, \infty)$  and a compact interval  $J \subset \mathbb{R}_+$ ,  $L^p(J, F)$  can be considered as a subspace of  $L^p(\mathbb{R}_+, F)$ . Hence,  $\Gamma_J f = \Gamma_{\mathbb{R}_+} f$  for  $f \in W(J, p)$  and  $E_J(t) f = E_{\mathbb{R}_+}(t) f$  for  $f \in X(J, p)$  and  $t \geq 0$ . In this case (jj)  $\Rightarrow$  (ii) is trivial.

Now, if  $p = \infty$ , then for any  $f \in W(J, \infty)$  there is  $\tilde{f} \in W(\mathbb{R}_+, \infty)$  such that  $\tilde{f} = f$  on  $J$  and  $\|f\|_{X(J, \infty)} = \|\tilde{f}\|_{X(\mathbb{R}_+, \infty)}$ . This implies, together with (jj), that (ii) holds.

(ii)  $\Rightarrow$  (jj) Take  $f \in C_c((0, \infty), W)$ , the space of continuous functions  $f : \mathbb{R}_+ \rightarrow W$  having compact support in  $(0, \infty)$ , and  $a, b > 0$  such that  $\operatorname{supp} f \subseteq [a, b]$  and set  $J := [a, b + T]$ . Then

$$\begin{aligned} \|\Gamma_{\mathbb{R}_+} E_{\mathbb{R}_+}(\cdot) f\|_{L^p(0, T; Y(\mathbb{R}_+, p))} &= \|\Gamma_J E_J(\cdot) f\|_{L^p(0, T; Y(J, p))} \\ &\leq \delta(T) \|f\|_{X(J, p)} \\ &= \delta(T) \|f\|_{X(\mathbb{R}_+, p)}. \end{aligned}$$

Therefore, Remark 3.6.a and the density of  $C_c((0, \infty), W)$  in  $W(\mathbb{R}_+, p)$ , for  $p \in [1, \infty]$ , allow us to obtain the result.  $\square$

#### 4. OUTPUT FUNCTION

In this section we show that if  $\mathcal{C}$  is an admissible observation family, we can give another interpretation of the state-output function

$$y(t, s, x) = C(t) x(t).$$

Here,  $x(\cdot)$  is the solution of the time-varying Cauchy problem

$$\begin{aligned} \dot{x}(t) &= A(t) x(t) \quad \text{for } t > s \geq 0, \\ x(s) &= x, \end{aligned} \tag{4.1}$$

with  $x \in X$  and unbounded operators  $(A(t), D(A(t)))$ ,  $t \in \mathbb{R}_+$ . Of course, the existence of the classical solution occurs if (4.1) is well-posed in the sense that there exists an evolution family  $(\phi(t, s))_{(t,s) \in \Delta_{\mathbb{R}_+}}$  solving (4.1), i.e., for  $s \geq 0$  and  $x \in D(A(s))$ , the function  $x(\cdot, s, x) = \phi(\cdot, s)x$  is differentiable in  $(s, \infty)$ ,  $x(t, s, x) \in D(A(t))$  for  $(t, s) \in \Delta_{\mathbb{R}_+}$ , and (4.1) holds.

In the sequel, we are concerned with the mild solution of (4.1), i.e., the continuous function

$$x(t, s, x) = \phi(t, s)x \quad \text{for } t \geq s \text{ and } x \in X.$$

In this section, we suppose that  $\phi$  is an exponentially bounded evolution family in both  $X$  and  $W$  and  $\mathcal{C} \in C_b(\mathbb{R}_+, \mathcal{L}_s(W, Y))$  is an a.o.f. for  $(W, X, Y, p, \phi)$ . Then, by Corollary 3.7, the multiplication operator  $\Gamma_{\mathbb{R}_+}$ , defined in Theorem 3.5, is an a.o.o. for  $(W(\mathbb{R}_+, p), X(\mathbb{R}_+, p), Y(\mathbb{R}_+, p), p, E_{\mathbb{R}_+}(\cdot))$ . As in [22] we define the *abstract linear observation system*  $((\mathcal{L}_T)_{T \geq 0}, E_{\mathbb{R}_+}(\cdot))$  on  $X(\mathbb{R}_+, p)$ , where  $\mathcal{L}_T$  denotes the unique continuous extension of the operator

$$W(\mathbb{R}_+, p) \ni f \mapsto \chi_{[0, T)}(\cdot) \Gamma_{\mathbb{R}_+} E_{\mathbb{R}_+}(\cdot) f \in L^p(\mathbb{R}_+, Y(\mathbb{R}_+, p)) \tag{4.2}$$

to all of  $X(\mathbb{R}_+, p)$  for  $T > 0$  and  $\mathcal{L}_0 = 0$ .

It is easy to verify that for any  $f \in X(\mathbb{R}_+, p)$ ,  $\mathcal{L}_T f$  converges in the Fréchet space  $L^p_{loc}(\mathbb{R}_+, Y(\mathbb{R}_+, p))$  equipped with the family of seminorms  $\|\cdot\|_{L^p(0, \tau; Y(\mathbb{R}_+, p))}$ ,  $\tau > 0$ , as  $T \rightarrow \infty$ . Let

$$\mathcal{L}_\infty f := \lim_{T \rightarrow \infty} \mathcal{L}_T f, \quad f \in X(\mathbb{R}_+, p).$$

By using the Banach-Steinhaus theorem generalized to Fréchet spaces (cf. [7], p. 195) we obtain that  $\mathcal{L}_\infty \in \mathcal{L}(X(\mathbb{R}_+, p), L^p_{loc}(\mathbb{R}_+, Y(\mathbb{R}_+, p)))$ .

Let  $(G_{\mathbb{R}_+}, D(G_{\mathbb{R}_+}))$  be the generator of the evolution semigroup  $E_{\mathbb{R}_+}(\cdot)$  on  $X(\mathbb{R}_+, p)$ . So, by Theorem 3.3 in [22], we obtain the following representation result.

**Theorem 4.1.** *Let  $\mathcal{C}$  and  $\phi$  be as above. Then there is a unique operator  $\Gamma_0 \in \mathcal{L}(D(G_{\mathbb{R}_+}), Y(\mathbb{R}_+, p))$ , where  $D(G_{\mathbb{R}_+})$  is endowed with the graph norm, such that*

$$(\mathcal{L}_\infty f)(t) = \Gamma_0 E_{\mathbb{R}_+}(t) f$$

for  $f \in D(G_{\mathbb{R}_+})$  and  $t \geq 0$ .

**Remark 4.2.** Note that  $\Gamma_0$  is an a.o.o. for

$(D(G_{\mathbb{R}_+}), X(\mathbb{R}_+, p), Y(\mathbb{R}_+, p), p, E_{\mathbb{R}_+}(\cdot))$  and it coincides with  $\Gamma_{\mathbb{R}_+}$  in  $D(G_{\mathbb{R}_+}) \cap W(\mathbb{R}_+, p)$ . Weiss [22] says that the operators  $\Gamma_0$  and  $\Gamma_{\mathbb{R}_+}$  are equivalent in the sense that they yield the same abstract linear observation system  $((\mathcal{L}_T)_{T \geq 0}, E_{\mathbb{R}_+}(\cdot))$ .

The fact that  $\Gamma_0$  is defined in  $D(G_{\mathbb{R}_+})$ , which is not the case for  $\Gamma_{\mathbb{R}_+}$ , and by using the property of an evolution family we give, via the representation result, Theorem 4.1, a more general interpretation of the output function associated to the time-varying evolution system, cf. Theorem 4.3. To this end, we introduce the output function of the non-autonomous Cauchy problem (4.1).

By  $H_{s,T}$ ,  $(T, s) \in \Delta_{\mathbb{R}_+}$ , we denote the unique continuous extension of the operator

$$W \ni x \longmapsto \chi_{[s,T]}(\cdot)C(\cdot)\phi(\cdot, s)x \quad (4.3)$$

to an element of  $\mathcal{L}(X, L^p(\mathbb{R}_+, Y))$ .

Moreover, it is clear that for any  $x \in X$  the sequence  $(H_{s,T}x)_{T \geq s}$  converges in the Fréchet space  $L_{loc}^p(\mathbb{R}_+, Y)$ , endowed with the seminorms  $\|\cdot\|_{L^p(0,\tau;Y)}$ ,  $\tau > 0$ , to some function  $H_{s,\infty}x$ , as  $T \rightarrow \infty$ . The obtained limit  $H_{s,\infty}x$  is called the *output function* of (4.1) associated to  $s$  and  $x$  and satisfies

$$(H_{s,\infty}x)(t) = C(t)\phi(t, s)x$$

for  $x \in W$  and  $t \geq s$ . This gives the observation of the system (4.1) for initial states  $x \in X$ . But this observation is only given almost everywhere since  $H_{s,\infty}x$  is an  $L_{loc}^p$ -function.

We can now establish the relationship between  $H_{s,\infty}$  and the abstract operator  $\Gamma_0$  given by Theorem 4.1.

**Theorem 4.3.** For  $x \in X$  and  $s \geq 0$  we have

$$\alpha(t)(H_{s,\infty}x)(t) = (\Gamma_0(\alpha(\cdot)\phi(\cdot, s)x))(t) \quad (4.4)$$

for all  $\alpha \in C_c^1(0, \infty)$  with  $\text{supp } \alpha \subset (s, \infty)$  and a.e.  $t \in [s, \infty)$ .

*Proof.* Let  $s \in \mathbb{R}_+$  and  $x \in X$ . We set

$$f_{s,x,\alpha} := \alpha(\cdot)\phi(\cdot, s)x \quad (4.5)$$

with  $\alpha \in C_c^1(0, \infty)$  and  $\text{supp } \alpha \subset (s, \infty)$ . Then, by the definition of the evolution semigroup  $E_{\mathbb{R}_+}(\cdot)$ , one can see that  $f_{s,x,\alpha} \in D(G_{\mathbb{R}_+})$  and  $G_{\mathbb{R}_+}f_{s,x,\alpha} = f_{s,x,-\alpha'}$ , where  $\alpha' := \frac{d}{dt}\alpha$ . In particular, if  $x \in W$ , then  $f_{s,x,\alpha} \in W(\mathbb{R}_+, p) \cap D(G_{\mathbb{R}_+})$ . Thus, by Theorem 4.1 and (4.2) we obtain

$$\begin{aligned} \Gamma_0 f_{s,x,\alpha} &= (\mathcal{L}_\infty f_{s,x,\alpha})(0) \\ &= \Gamma_{\mathbb{R}_+} f_{s,x,\alpha} \\ &= \alpha(\cdot)C(\cdot)\phi(\cdot, s)x \\ &= \alpha(\cdot)H_{s,\infty}x \end{aligned} \quad (4.6)$$

for  $x \in W$ . Let  $x \in X$ . Since  $W$  is dense in  $X$ , there is a sequence  $(x_n) \subset W$  such that  $\lim_{n \rightarrow \infty} \|x_n - x\| = 0$ . Hence,

$$\lim_{n \rightarrow \infty} \|f_{s,x_n,\alpha} - f_{s,x,\alpha}\|_{X(\mathbb{R}_+,p)} = \lim_{n \rightarrow \infty} \|f_{s,x_n,\alpha'} - f_{s,x,\alpha'}\|_{X(\mathbb{R}_+,p)} = 0.$$

Therefore,

$$\lim_{n \rightarrow \infty} \|f_{s,x_n,\alpha} - f_{s,x,\alpha}\|_{D(G_{\mathbb{R}_+})} = 0.$$

By using the facts that  $H_{s,\infty} \in \mathcal{L}(X, L_{loc}^p(\mathbb{R}_+, Y))$  and  $\Gamma_0 \in \mathcal{L}(D(G_{\mathbb{R}_+}), Y(\mathbb{R}_+, p))$ , we see that (4.4) follows from (4.6).  $\square$

**Remark 4.4.** *In the situation of Theorem 4.3, assume that for some  $x \in X$  and  $s \geq 0$  the function  $f_{s,x,\alpha} \in W(\mathbb{R}_+, p)$  for all  $\alpha \in C_c^1(0, \infty)$  with  $\text{supp } \alpha \subset (s, \infty)$ . Then we have*

$$(H_{s,\infty}x)(t) = C(t)\phi(t, s)x \text{ for a.e. } t \geq s.$$

Hence, the output function conserves its expression as an  $L_{loc}^p$ -function even if  $x \notin W$ .

## 5. AN APPLICATION

We consider the (formal) differential operator

$$A(t, x, D) := \sum_{i,j=1}^n a_{ij}(t, x) D_i D_j + \sum_{i=1}^n b_i(t, x) D_i$$

for  $t \geq 0$  and  $x \in \Omega$ , where  $\Omega$  is a bounded subset of  $\mathbb{R}^n$  with smooth boundary  $\partial\Omega$ . On the coefficients of  $A(t, x, D)$  we impose

$$(H1) \quad a_{ij} = a_{ji}, \quad b_i \in C_b^1(\mathbb{R}_+, C^2(\bar{\Omega}, \mathbb{R})).$$

$$(H2) \quad \sum_{i,j=1}^n a_{ij}(t, x) \xi_i \xi_j \geq \delta_0 |\xi|^2 \text{ for } \xi \in \mathbb{R}^n, t \geq 0, x \in \bar{\Omega} \text{ and a constant } \delta_0 > 0.$$

Let  $1 < r < \infty$ . We define the  $L^r$ -realization  $(A_D(t), W)$ ,  $t \geq 0$ , of  $A(t, \cdot, D)$  with Dirichlet boundary conditions by

$$A_D(t)u := A(t, \cdot, D)u \quad \text{for } u \in W := W^{2,r}(\Omega) \cap W_0^{1,r}(\Omega).$$

Then the non-autonomous Cauchy problem

$$\begin{cases} u'(t) &= A(t, \cdot, D)u(t), \quad t \geq s \geq 0, \\ u(s) &= u_s \end{cases} \quad (5.1)$$

is well-posed on the space  $W$  (cf. [16], Lemma 7.6.1 and Theorem 5.6.1). The solution is given by a positive evolution family  $(\phi(t, s))_{(t,s) \in \Delta_{\mathbb{R}_+}}$  on  $L^r(\Omega)$  satisfying an upper Gaussian estimate, that is, there are constants  $N, a > 0$  such that

$$0 \leq \phi(t, s)u \leq NG(a(t-s))u, \quad (t, s) \in \Delta_{\mathbb{R}_+}, \quad (5.2)$$

for  $0 \leq u \in L^r(\Omega)$ , where  $G(0) := I$  and  $G(t)u := \chi_\Omega(\cdot)(K_t * u)$ ,  $t > 0$ , with  $K_t(x) := (4\pi t)^{-n/2} e^{-|x|^2/4t}$  for  $x \in \mathbb{R}^n$  and  $u \in L^r(\Omega)$  is extended by 0 to  $\mathbb{R}^n$ . Moreover, we have

$$\phi(t, s)L^r(\Omega) \subset W \text{ and } \|A_D(t)\phi(t, s)\| \leq \frac{\tilde{M}}{t-s}, \quad t > s \geq 0, \quad (5.3)$$

for some constant  $\tilde{M} > 0$  (cf. [16], Theorem 5.6.1).

We do not attempt to treat the most general situation. For more examples we refer to the book of H. Tanabe [21] and the references therein.

We suppose now that  $\frac{n}{2} < r < \infty$  and consider the observation operator

$$C(t)u := u(c(t)) \quad \text{for } u \in W,$$

where  $c(\cdot) \in C(\mathbb{R}_+, \overline{\Omega})$ . Since  $r > \frac{n}{2}$ , it follows that  $W^{2,r}(\Omega)$  (and hence  $(W, \|\cdot\|_{D(A_D(t))})$ ) is continuously embedded in  $C(\overline{\Omega})$  (cf. [10], Corollary 7.7.11). Therefore,  $C(\cdot) \in C_b(\mathbb{R}_+, \mathcal{L}_s(W, \mathbb{R}))$ .

The following result shows that  $C(\cdot)$  is an a.o.f. for  $(W, L^r(\Omega), \mathbb{R}, p, \phi)$  for some  $p \in (1, \infty)$  and gives the expression of the output function  $H_{s,\infty}$  of (5.1) associated to  $s$  and  $u_s$ .

**Proposition 5.1.** *Assume (H1)-(H2) and  $\frac{n}{2} < r < \infty$ . If  $p \in [1, \frac{2r}{n})$ , then  $C(\cdot)$  is an a.o.f. for  $(W, L^r(\Omega), \mathbb{R}, p, \phi)$ . Moreover the output function  $H_{s,\infty}$  of (5.1) is given by*

$$(H_{s,\infty}u_s)(t) = (\phi(t, s)u_s)(c(t)) \quad (5.4)$$

for  $u_s \in L^r(\Omega)$  and a.e.  $t \geq s$ .

*Proof.* By an easy computation we see that

$$\|K_t\|_{L^{r'}} = (4\pi t)^{-\frac{n}{2r}}(r')^{-\frac{n}{2r'}} =: \tilde{M}t^{-\frac{n}{2r}},$$

where  $\frac{1}{r} + \frac{1}{r'} = 1$  and  $\tilde{M} > 0$  is a constant depending on  $n$  and  $r$ . Using this identity, Young's inequality, and (5.2), we obtain

$$\begin{aligned} \|C(\cdot)\phi(\cdot, s)u\|_{L^p(s,T)}^p &= \int_s^T |(\phi(t, s)u)(c(t))|^p dt \\ &\leq N^p \int_s^T |(K_{a(t-s)} * |u|)(c(t))|^p dt \\ &\leq N^p \int_s^T \|K_{a(t-s)} * |u|\|_\infty^p dt \\ &\leq N^p \int_s^T \|K_{a(t-s)}\|_{L^{r'}}^p dt \|u\|_{L^r}^p \\ &= N^p \tilde{M}^p \left( \int_s^T (t-s)^{-\frac{np}{2r}} dt \right) a^{-\frac{np}{2r}} \|u\|_{L^r}^p \end{aligned}$$

for  $u \in W$  and  $(T, s) \in \Delta_{\mathbb{R}_+}$ . Since  $p \in [1, \frac{2r}{n})$ , it follows that

$$\|C(\cdot)\phi(\cdot, s)u\|_{L^p(s,T)} \leq \gamma(T-s)\|u\|_{L^r}$$

for  $u \in W$  and  $(T, s) \in \Delta_{\mathbb{R}_+}$ , where

$$\gamma(t) := \frac{N\tilde{M}a^{-\frac{n}{2r}}}{(1 - \frac{np}{2r})^{\frac{1}{p}}} \left( t^{1-\frac{np}{2r}} \right)^{\frac{1}{p}}, \quad t \geq 0.$$

This implies that  $C(\cdot)$  is an a.o.f. for  $(W, L^r(\Omega), \mathbb{R}, p, \phi)$ .

It now remains to give the expression of the output function  $H_{s,\infty}$  of (5.1) with the observation family  $(C(t))_{t \geq 0}$ . From Theorem II.4.4.1 in [1] it follows that  $\phi$  is an exponentially bounded evolution family in both  $L^r(\Omega)$  and  $W$ . We are then in the situation of Theorem 4.3. Further, it follows from (5.3) that the functions

$$f_{s,u_s,\alpha} := \alpha(\cdot)\phi(\cdot, s)u_s$$

belong to  $L^p(\mathbb{R}_+, W)$  ( $= W(\mathbb{R}_+, p)$ ) for  $u_s \in L^r(\Omega)$ ,  $s \geq 0$ , and  $\alpha \in C_c^1(0, \infty)$  with  $\text{supp } \alpha \subset (s, \infty)$ . Hence one can apply Remark 4.4 and we obtain (5.4).  $\square$



**Remark 5.2.** From (5.4) we see that the output function  $H_{s,\infty}$  is everywhere equal to a continuous function on  $(s, \infty)$  and is continuously dependent on the initial state  $u_s \in L^r(\Omega)$  when it is considered as an  $L^p_{loc}$ -function for  $r \geq \frac{n}{2}$  and  $p \in [1, \frac{2r}{n})$ .

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# Numerical Solution of Linear Multi-Term Initial Value Problems of Fractional Order

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## Abstract

In this paper, a new algorithm for the numerical solution of the initial value problems for general linear multi-term differential equations of fractional order with constant coefficients and fractional derivatives defined in the Caputo sense is presented. The algorithm essentially uses some ideas from the convolution quadrature and discretized operational calculus. Another basic element of the method is the formulas for analytical solution of the problem under consideration given in terms of the Mittag-Leffler type functions. Error estimates and numerical examples are presented. Special attention is given to the comparison of the numerical results obtained by the new algorithm with those found by other known methods.

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*Key Words and Phrases:* fractional differential equations, convolution quadrature, Caputo fractional derivative, Mittag-Leffler type functions

## 1 Introduction

In this paper, we propose a method for the numerical evaluation of the solution of the initial value problem for the general linear multi-term equation of fractional

order of the type

$$\begin{cases} (D_*^\mu x)(t) - \sum_{i=1}^\nu \lambda_i (D_*^{\mu_i} x)(t) = f(t), & 0 < t \leq T < \infty, \\ x^{(k)}(0) = c_k \in \mathbb{R}, & k = 0, \dots, m-1, \quad m-1 < \mu \leq m \in \mathbb{N}, \end{cases} \quad (1)$$

where  $\mu > \mu_1 > \dots > \mu_\nu \geq 0$ ,  $m_i - 1 < \mu_i \leq m_i$ ,  $m_i \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ ,  $\lambda_i \in \mathbb{R}$ ,  $i = 1, \dots, \nu$ ,  $f$  is a given function defined on the interval  $]0, T]$ ,  $x$  is an unknown function and the fractional derivatives are defined in the Caputo sense (see e.g. [21, 22])

$$(D_*^\alpha x)(t) := \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} x^{(n)}(\tau) d\tau, \quad (2)$$

$$n-1 < \alpha < n, \quad n \in \mathbb{N}, \quad 0 < t \leq T < \infty;$$

$$(D_*^\alpha x)(t) := x^{(\alpha)}(t), \quad \alpha \in \mathbb{N}_0, \quad 0 < t \leq T < \infty.$$

This integro-differential operator has apparently first been used by Caputo [2] to introduce some new models for dissipation in inelastic solids and independently by Rabotnov [25] in the modeling of viscoelastic materials; it is related but not identical to the Riemann-Liouville differential operator

$$(D^\alpha x)(t) := \frac{d^n}{dt^n} \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} x(\tau) d\tau$$

(where  $n$  is related to  $\alpha$  in the same way as in the Caputo case) that is more common in pure mathematics (see e.g. [24, 26]).

There are some important applications for equations of this type known in engineering and other fields. A prominent example is the Bagley-Torvik equation

$$A(D_*^2 x)(t) + B(D_*^{3/2} x)(t) + C(D_*^0 x)(t) = f(t), \quad (3)$$

typically used with homogeneous initial conditions  $x(0) = x'(0) = 0$ , that describes the motion of a rigid plate immersed in a Newtonian viscous fluid, cf. [24, §8.3.2]. Note that traditionally the Bagley-Torvik equation is formulated with Riemann-Liouville differential operators rather than with Caputo derivatives, but as remarked in [24] under these initial conditions the two problems are equivalent. However, from [6] we take the information that the formulation in terms of Caputo operators is much more useful when other physical systems requiring inhomogeneous initial conditions are to be modeled. A further example is Babenko's model describing the process of solution of a gas in a fluid [24, §8.3.3], which leads to the equation

$$F(t)(D_*^1 x)(t) + G(t)(D_*^{1/2} x)(t) + (D_*^0 x)(t) = -1.$$

In some important special cases we may assume the functions  $F$  and  $G$  to be constant; then the problem reduces to a special case of (1). An interested reader is also advised to consult a recent report by Freed et al. [13] where the popular

one-dimensional fractional-order viscoelastic material models were extended into their three-dimensional equivalents for finitely deforming continua.

For more information on the Caputo derivative and its utility we refer to [15, 22]; other recent general references to fractional differential equations and fractional calculus may be found in [1, 24, 26].

Among the main components of the algorithm presented in the paper are the formulas for analytical solution of the problem (1) obtained by Gorenflo and Luchko [21] using an operational calculus for the Caputo fractional derivative in an appropriate space of functions. The solution given in [21] is in terms of the Mittag-Leffler type functions. In the general case, this function is defined as a multiple power series with fractional exponents; if  $\nu = 1$  it reduces to the well known Mittag-Leffler function  $E_{\alpha,\beta}$ . Whereas an algorithm for numerical evaluation of the Mittag-Leffler function  $E_{\alpha,\beta}$  was presented in [14], the numerical evaluation of the general Mittag-Leffler type function is still an open problem; we provide a partial solution to this problem in the next section. Other methods for obtaining the analytical solutions of special cases of (1) are discussed, e.g., in [24, Chapter 5].

In the paper, we combine the analytical results from [21] and the convolution quadrature and discretized operational calculus developed in [19, 20] that leads to a new algorithm for numerical solution of the problem (1). Under suitable restrictions on the function  $f$  from the right-hand side of (1) the proposed approximation is convergent of the order of the multistep method underlying for the convolution quadrature method. Note that a similar method has been proposed and investigated in [6] for the special case of the Bagley-Torvik equation, but there the proofs explicitly rely on commensurability assumptions on the orders of the differential operators. We are now in a position to generalize these results to a much larger class of equations, and in particular we avoid such assumptions.

The remainder of the paper is organized as follows. In the second section, the case of the homogeneous problem corresponding to (1) is considered. In the third section, an algorithm for numerical solution of the general problem (1) is presented. The fourth section gives an overview of related work. Finally, in the fifth section some numerical examples are discussed.

## 2 Homogeneous differential equations of fractional order

First, for reader's convenience, we present some results from [21] concerning the analytical solution of the initial value problem (1) that are used in the further discussions. Following [10] and [21] we introduce the space  $C_\alpha^m$  of functions where the solution of (1) is searched:

**Definition 1** *A real or complex-valued function  $f(t)$ ,  $t > 0$ , is said to be in the space  $C_\alpha$ ,  $\alpha \in \mathbb{R}$ , if there exists a real number  $p$ ,  $p > \alpha$ , such that  $f(t) = t^p \tilde{f}(t)$ ,  $\tilde{f} \in C[0, \infty)$ . A function  $f$  is in the space  $C_\alpha^m$ ,  $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$  if  $f^{(m)} \in C_\alpha$ .*

The solution of the initial value problem (1) in the space  $C_{-1}^m$ , i.e. in the classical sense, is given by the following result [21]:

**Theorem 1** *The initial value problem (1) with a function  $f$  represented in the form  $f(t) = t^{\gamma-1} \tilde{f}(t)$  with  $\tilde{f} \in C[0, T]$  and  $\gamma > 0$  if  $\mu \in \mathbb{N}$  and  $\gamma \geq m - \mu$  if  $\mu \notin \mathbb{N}$  ( $m - 1 < \mu \leq m \in \mathbb{N}$ ) has a solution in the space  $C_{-1}^m$ , unique in this space, given by the formula*

$$x(t) = x_{\sim f}(t) + \sum_{k=0}^{m-1} c_k u_k(t), \quad t \geq 0. \quad (4)$$

Here

$$x_{\sim f}(t) = \int_0^t E(\mu; \tau) f(t - \tau) d\tau \quad (5)$$

is a solution of the problem (1) with zero initial conditions, and the system of functions

$$u_k(t) = \frac{t^k}{k!} + \sum_{i=l_k+1}^{\nu} \lambda_i E(k+1+\mu-\mu_i; t), \quad k = 0, \dots, m-1, \quad (6)$$

fulfills the initial conditions  $u_k^{(l)}(0) = \delta_{kl}$ ,  $k, l = 0, \dots, m-1$ ,  $\delta_{kl}$  is the Kronecker symbol. The function

$$E(\beta; t) = t^{\beta-1} E_{(\mu-\mu_1, \dots, \mu-\mu_\nu), \beta}(\lambda_1 t^{\mu-\mu_1}, \dots, \lambda_\nu t^{\mu-\mu_\nu}), \quad \beta > 0, \quad (7)$$

is given in terms of the multivariate Mittag-Leffler function

$$E_{(\alpha_1, \dots, \alpha_\nu), \beta}(z_1, \dots, z_\nu) := \sum_{k=0}^{\infty} \sum_{\substack{l_1 + \dots + l_\nu = k \\ l_1 \geq 0, \dots, l_\nu \geq 0}} \frac{k!}{l_1! \times \dots \times l_\nu!} \frac{\prod_{i=1}^{\nu} z_i^{l_i}}{\Gamma(\beta + \sum_{i=1}^{\nu} \alpha_i l_i)}$$

and the natural numbers  $l_k$ ,  $k = 0, \dots, m-1$ , are determined from the condition

$$\begin{cases} m_{l_k} \geq k+1, \\ m_{l_k+1} \leq k. \end{cases} \quad (8)$$

In the case  $m_1 \leq k$ , we set  $l_k := 0$ , and if  $m_\nu \geq k+1$ ,  $i = 0, \dots, m-1$ , then  $l_k := \nu$ .

Gorenflo and Luchko [21] proved Theorem 1 in the case  $\mu \notin \mathbb{N}$  under the assumption that the function  $f$  lies in the space  $C_{-1}^1$  which is more restrictive than our condition. But their proof is still valid (with some small modifications) for the initial value problem (1) with the right-hand side taken in the form  $f(t) = t^{\gamma-1} \tilde{f}(t)$  with  $\tilde{f} \in C[0, T]$  and  $\gamma \geq m - \mu$ .

In this section we consider the homogeneous problem (1), i.e., the case  $f(t) \equiv 0$ ,  $0 < t \leq T < \infty$ . It follows from Theorem 1 that the analytical solution of the

homogeneous problem (1) is given in terms of the Mittag-Leffler type function (7).

Except for the case of the Mittag-Leffler function  $E_{\alpha,\beta}$  (that corresponds to  $\nu = 1$  in (7)) considered in detail in [14], no algorithms are known for the numerical evaluation of the multivariate Mittag-Leffler function. In this section, we describe a method for evaluation of a particular case of the general multivariate Mittag-Leffler function, namely, of the function (7). The method uses a formula for the Laplace transform of (7) obtained in [21] and the convolution quadrature method presented in [19].

Let

$$F(\beta; s) := \frac{s^{\mu-\beta}}{s^{\mu} - \sum_{i=1}^{\nu} \lambda_i s^{\mu_i}}, \quad \beta > 0, \quad (9)$$

where the complex variable  $s$  is in the complex plane with a cut along the negative real semi-axis and the main branch of the power function  $P_{\gamma}(s) = s^{\gamma}$  is chosen.

The convolution quadrature method discussed in [19] applies a linear multistep method to an initial value problem for the differential equation  $y'(t) = \lambda y(t) + f(t)$  with zero initial conditions  $y(0) = 0$ . Let

$$\delta(\zeta) := \sum_{j=0}^{\infty} \delta_j \zeta^j \quad (10)$$

be the quotient of its generating polynomials.

We shall always assume in the further discussions that the linear multistep method under consideration is  $A(\alpha)$ -stable, stable in a neighbourhood of infinity, strongly zero-stable and consistent of order  $p$ . In particular, this is the case for the backward differentiation formulas given by

$$\delta(\zeta) = \sum_{j=1}^p \frac{1}{j} (1 - \zeta)^j, \quad p = 1, 2, \dots \quad (11)$$

with  $\alpha = 90^\circ, 90^\circ, 88^\circ, 73^\circ, 51^\circ, 18^\circ$  for  $p = 1, \dots, 6$ , respectively.

Let  $h$  be the stepsize of our approximation and

$$G_h(T) := \{t_n = nh \mid n = 1, 2, \dots, N(h), \quad N(h) = [T/h]\} \quad (12)$$

be the equispaced grid.

Following the scheme of Lubich [19] for approximation of the inverse Laplace transform we define an approximation  $E_h$  of the Mittag-Leffler type function (7) at the point  $t_n = nh$  of the grid  $G_h(T)$  as

$$E_h(\beta; t_n) := \omega_n(h)/h, \quad (13)$$

where  $\omega_j(h)$ ,  $j = 0, 1, \dots, N(h)$  are the coefficients of the power series

$$F(\beta; \delta(\zeta)/h) = \sum_{j=0}^{\infty} \omega_j(h) \zeta^j. \quad (14)$$

Here  $\delta(\zeta) = \sum_{j=0}^{\infty} \delta_j \zeta^j$  is the quotient of the generating polynomials of a linear multistep method and the function  $F$  is given by (9).

For the error of this approximation we have the following result:

**Theorem 2** *Let  $E_h$  be the approximation (13) of the Mittag-Leffler type function (7) at the point  $t_N = T$ . Then*

$$|E_h(\beta; T) - E(\beta; T)| = O(h^p), \quad h \rightarrow 0, \quad (15)$$

where a linear multistep method used to generate the approximation (13) is consistent of order  $p$ .

*Proof.* To prove the theorem we use the general result from [19] about numerical evaluation of a function given as an inverse Laplace transform of another function that possesses certain properties. In our special case we use the results from [21] showing that the Mittag-Leffler type function (7) can be represented as the inverse Laplace transform of the function  $F$  given by (9). Let us introduce the function

$$\psi(\beta; s) := \frac{1}{F(\beta, s)} = s^{\beta-\mu} \left( s^{\mu} - \sum_{i=1}^{\nu} \lambda_i s^{\mu_i} \right),$$

where the complex variable  $s$  is in the complex plane with a cut along the negative real semi-axis and the main branch of the power functions is chosen. Since  $\beta > 0$  and  $\mu > \mu_1 > \dots > \mu_{\nu} \geq 0$  we have the estimate

$$|\psi(\beta; s)| \geq M |s|^{\beta}, \quad |s| > s_0(M) > 0, \quad 1 > M > 0.$$

It means that all zeros of the function  $\psi$  are inside of a circle with the radius  $s_0(M)$ . Therefore, for any positive  $\alpha$  (we recall that the linear multistep method used in our algorithm is  $A(\alpha)$ -stable) we can choose such positive numbers  $\phi < \pi/2$  and  $c$  that the function  $F = 1/\psi$  is analytic in a sector  $|\arg(s - c)| < \pi - \phi$  with  $\phi < \alpha$  and satisfies there the following estimate

$$|F(\beta; s)| \leq C |s|^{-\beta}, \quad C = 1/M < \infty, \quad \beta > 0.$$

It follows from the properties of the linear multistep method used in our algorithm that  $\delta_0 > 0$ . Consequently, for sufficiently small  $h$  and  $\zeta$  the expression  $\delta(\zeta)/h$  is in the domain of analyticity of  $F$  and the power series (14) is well defined.

We are now in a position to apply Theorem 4.1 from [19] concerning the approximation of a function represented as inverse Laplace transform of another function that leads to the estimate

$$|E_h(\beta; t_n) - E(\beta; t_n)| \leq C n^{\beta-1-p} h^{\beta-1}. \quad (16)$$

Here the constant  $C$  does not depend on  $h \in (0, h_0]$ , for a sufficiently small  $h_0$ , and on  $n \in \{1, 2, \dots, N(h)\}$ ;  $E_h$  is the approximation (13) of the Mittag-Leffler type function (7) at the point  $t = t_n$ ,  $t_n \in G_h(T)$ . Specializing the inequality (16) for the case  $n = N = T/h$  we arrive at the estimate (15). ■



**Remark 1** The definition (7) of the Mittag-Leffler type function implies that

$$E(\beta; 0) = \begin{cases} 0, & \beta > 1, \\ 1, & \beta = 1, \\ \infty, & 0 < \beta < 1. \end{cases}$$

In the first two cases we can additionally define the approximation  $E_h$  at the point  $t = t_0 = 0$  by 0 and 1, respectively.

Now we consider the homogeneous problem (1) defining an approximation  $x_h$  of its solution  $x$  at the point  $t_n = nh$  of the grid  $G_h(T)$  as follows:

$$x_h(t_n) := \sum_{k=0}^{m-1} c_k \left( \frac{(t_n)^k}{k!} + u_h(k; t_n) \right), \quad n = 1, \dots, N(T), \quad (17)$$

where

$$u_h(k; t_n) := \omega_n(k; h)/h. \quad (18)$$

The numbers  $\omega_j(k; h)$ ,  $k = 0, 1, \dots, m-1$ ,  $j = 0, 1, \dots, N(T)$  are the coefficients of the power series

$$F_k(\delta(\zeta)/h) = \sum_{j=0}^{\infty} \omega_j(k; h) \zeta^j, \quad (19)$$

where  $\delta(\zeta) = \sum_{j=0}^{\infty} \delta_j \zeta^j$  is the quotient of the generating polynomials of the underlying linear multistep method,

$$F_k(s) := \frac{\sum_{i=l_k+1}^{\nu} \lambda_i s^{\mu_i - k - 1}}{s^{\mu} - \sum_{i=1}^{\nu} \lambda_i s^{\mu_i}}, \quad (20)$$

and the natural numbers  $l_k$ ,  $k = 0, \dots, m-1$  are determined as in Theorem 1.

The error of the approximation is given by the following result:

**Theorem 3** Let  $x_h$  be the approximation (17) of the solution  $x$  of the homogeneous equation (1) at the point  $t_N = T$ . Then

$$|x_h(T) - x(T)| = O(h^p), \quad h \rightarrow 0, \quad (21)$$

where a linear multistep method used to generate the approximation (17) is consistent of order  $p$ .

*Proof.* According to Theorem 1 the analytical solution of the homogeneous equation (1) can be represented in the form

$$x(t) = \sum_{k=0}^{m-1} c_k \left( \frac{t^k}{k!} + u_k(t) \right),$$

where

$$u_k(t) = \sum_{i=l_k+1}^{\nu} \lambda_i E(\mu - \mu_i + k + 1; t), \quad k = 0, \dots, m-1,$$

$E$  being the Mittag-Leffler type function (7). Moreover, it was shown in [21] that the functions  $u_k$ ,  $k = 0, 1, \dots, m-1$  can be represented as the inverse Laplace transform of the function  $F_k$  given by (20). Noting that

$$F_k(s) = \sum_{i=l_k+1}^{\nu} \lambda_i F(\mu - \mu_i + k + 1; s),$$

where the function  $F$  is defined by (9) and applying the inequalities (16) we arrive, for sufficiently small  $h_0$ , at the estimates

$$|x_h(t_n) - x(t_n)| \leq C n^{\beta-1-p} h^{\beta-1}, \quad \beta := \begin{cases} \mu - \mu_1 + 1, & t_n \leq 1, \\ \mu - \mu_\nu + m, & t_n > 1. \end{cases} \quad (22)$$

Here the constant  $C$  does not depend on  $h \in (0, h_0]$  and on  $n \in \{1, 2, \dots, N(h)\}$ . In particular, for  $n = N = T/h$  the estimate (22) reduces to (21). ■

**Remark 2** For  $n = 0$ , we define the approximation (17) as  $x_h(0) := c_0$ .

**Remark 3** To compute  $\omega_j(h)$ ,  $j = 0, 1, \dots, N(T)$  in (14) and  $\omega_j(k; h)$ ,  $k = 0, \dots, m-1$ ,  $j = 0, 1, \dots, N(T)$  in (19) the formal power series method and FFT technique can be used among other methods (see e.g. [17, 20]).

### 3 Inhomogeneous differential equations of fractional order

In this section we first consider the inhomogeneous problem (1) with zero initial conditions and then the general case of the initial value problem (1). We shall assume that the right-hand side of the equation (1), i.e. the function  $f$ , can be represented in the form  $f(t) = t^{\gamma-1} \tilde{f}(t)$  with  $\tilde{f} \in C^p[0, T]$ ,  $p \geq 1$  and  $\gamma > 0$  if  $\mu \in \mathbb{N}$  and  $\gamma \geq m - \mu$  if  $\mu \notin \mathbb{N}$ . Let us begin with the problem with zero initial conditions, i.e.  $c_k = 0$ ,  $k = 0, \dots, m-1$ , in (1). In this case we can apply Theorem 1 to represent the analytical solution  $x_{\sim f}$  of the problem (1) in the form

$$x(t)_{\sim f} = \int_0^t E(\mu; \tau) f(t - \tau) d\tau, \quad (23)$$

where the Mittag-Leffler type function  $E$  is defined by (7). The function  $E$  can be alternatively represented as the inverse Laplace transform of the function  $F$  given by (9) with  $\beta = \mu$  (see [21]).

Using the same arguments as in the proof of Theorem 2 we can show that there exist positive numbers  $\phi < \pi/2$  and  $c$  such that the function  $F$  is analytic in a sector  $|\arg(s - c)| < \pi - \phi$  with  $\phi < \min\{\alpha, \pi/2\}$  and satisfies there the estimate

$$|F(\mu; s)| \leq C |s|^{-\mu}, \quad C < \infty.$$

The last estimate allows us to use the numerical scheme proposed by Lubich [19] to approximate the convolution integral (23) at the point  $t_n \in G_h(T)$ :

$$x_h(t_n)_{\sim f} := \sum_{j=0}^n \omega_j(h) f(t_{n-j}), \quad n = 1, \dots, N(h), \quad (24)$$

where  $\omega_j(h)$ ,  $j = 0, 1, \dots, N(h)$ , are given by (14) with  $\beta$  substituted by  $\mu$ . If  $f(t) = t^{\gamma-1} \tilde{f}(t)$  with  $0 < \gamma < 1$ , the approximation (24) is to be understood as

$$x_h(t_n)_{\sim f} := \sum_{j=0}^{n-1} \omega_j(h) f(t_{n-j}), \quad n = 1, \dots, N(T). \quad (25)$$

Applying Theorem 5.2 from [19] with the order  $p$  of the multistep method underlying for the used convolution quadrature method we immediately get the estimates

$$|x_h(t_n)_{\sim f} - x(t_n)_{\sim f}| \leq \begin{cases} C n^{\mu-1} h^{\mu-1+\gamma}, & 0 < \gamma \leq p, \\ C n^{\mu-1+\gamma-p} h^{\mu-1+\gamma}, & p \leq \gamma. \end{cases} \quad (26)$$

Here, for sufficiently small  $h_0$ , the constant  $C$  does not depend on  $h \in (0, h_0]$  and  $n \in \{1, 2, \dots, N(h)\}$ . In particular, if  $n = N = T/h$ , we arrive at the following result:

**Theorem 4** *Let  $x_h(t_n)_{\sim f}$  be the approximation (24) (or, for  $0 < \gamma < 1$ , the approximation (25)) at the point  $t_n = T$  of the solution  $x_{\sim f}$  of the inhomogeneous problem (1) with zero initial conditions and the right-hand side in the form  $f(t) = t^{\gamma-1} \tilde{f}(t)$ ,  $\tilde{f} \in C^p[0, T]$ ,  $p \geq 1$ ,  $\gamma > 0$  if  $\mu \in \mathbb{N}$  and  $\gamma \geq m - \mu$  if  $\mu \notin \mathbb{N}$ . Then*

$$|x_h(T)_{\sim f} - x(T)_{\sim f}| = O(h^q), \quad h \rightarrow 0, \quad q = \min\{\gamma, p\}, \quad (27)$$

where a linear multistep method used to generate the approximation (24) (or, for  $0 < \gamma < 1$ , the approximation (25)) is consistent of order  $p$ .

**Remark 4** *Because we consider the problem (1) with zero initial conditions, the approximation (24) can be additionally defined at the point  $t_0 = 0$  by  $x_h(0) := 0$ .*

If  $\gamma < p$ , the convergence order  $p$  can be restored in theory by a simple modification of the approximation (24):

$$\tilde{x}_h(t_n)_{\sim f} = x_h(t_n)_{\sim f} + \sum_{j=j_0}^{q-1} w_{nj}(h) f(t_j), \quad q-1 < p-\gamma \leq q \in \mathbb{N}, \quad (28)$$

where  $j_0 = 1$  if  $0 < \gamma < 1$  or  $j_0 = 0$  if  $1 \leq \gamma$  and the correction quadrature weights  $w_{nj}(h)$ ,  $j = j_0, \dots, q-1$  are determined, for each  $n = 1, 2, \dots, N(T)$ , from the generalized Vandermonde system

$$\sum_{j=j_0}^{q-1} w_{nj}(h) (t_j)^{k+\gamma-1} = x(t_n)_{\sim t^{k+\gamma-1}} - x_h(t_n)_{\sim t^{k+\gamma-1}}, \quad k = 0, 1, \dots, q-1. \quad (29)$$

In practice, there are a number of substantial numerical difficulties associated with this system; we refer to [5] and [27] for a discussion of the problems and potential solutions.

The convergence of the approximation (28) is given by the following result:

**Theorem 5** *Let  $\tilde{x}_h(t_n)_{\sim f}$  be the approximation (28) of the solution  $x_{\sim f}$  of the inhomogeneous problem (1) with zero initial conditions and the right-hand side in the form  $f(t) = t^{\gamma-1}\tilde{f}(t)$ ,  $\tilde{f} \in C^p[0, T]$ ,  $p \geq 1$ ,  $\gamma > 0$  if  $\mu \in \mathbb{N}$  and  $\gamma \geq m - \mu$  if  $\mu \notin \mathbb{N}$  at the point  $t_n \in G_h(T)$ . Then, for sufficiently small  $h_0$ , there exists a constant  $C$  not depending on  $h \in (0, h_0]$  and  $n \in \{1, 2, \dots, N(h)\}$  such that the estimates*

$$|\tilde{x}_h(t_n)_{\sim f} - x(t_n)_{\sim f}| \leq \begin{cases} C n^{\mu-1} h^{\mu-1+p}, & t_n \leq 1, \\ C n^{\mu-1+q+\gamma-p} h^{\mu-1+q+\gamma}, & t_n > 1, \end{cases} \quad (30)$$

hold. In particular,

$$|\tilde{x}_h(T)_{\sim f} - x(T)_{\sim f}| = O(h^p), \quad h \rightarrow 0, \quad (31)$$

where a linear multistep method used to generate the approximation (28) is consistent of order  $p$ .

*Proof.* The Taylor expansion of  $\tilde{f} \in C^p[0, T]$  at the point  $t = 0$  gives us the representation

$$f(t) = t^{\gamma-1} \sum_{k=0}^{q-1} \frac{\tilde{f}^{(k)}(0)}{k!} t^k + r(t), \quad (32)$$

where

$$r(t) = \frac{t^{\gamma-1}}{(q-1)!} (P_{q-1} * \tilde{f}^{(q)})(t), \quad P_{q-1}(t) := t^{q-1},$$

and the number  $q \in \mathbb{N}$  is defined to satisfy the inequalities  $q-1 < p-\gamma \leq q$ . Since  $\tilde{f} \in C^p[0, T]$ , we have  $\tilde{f}^{(q)} \in C^{p-q}[0, T]$  and the remainder  $r$  can be represented in the form

$$r(t) = t^{q+\gamma-1} \phi(t), \quad \phi \in C^p[0, T]. \quad (33)$$

According to (29) the quadrature formula (28) is exact for the products of the power function  $P_{\gamma-1}$  and the polynomials up to degree  $q-1$ :

$$\tilde{x}_h(t_n)_{\sim t^{k+\gamma-1}} - x(t_n)_{\sim t^{k+\gamma-1}} = 0, \quad k = 0, 1, \dots, q-1.$$

Together with the representation (32) it gives us the relation

$$\begin{aligned} \tilde{x}_h(t_n)_{\sim f} - x(t_n)_{\sim f} &= \tilde{x}_h(t_n)_{\sim r} - x(t_n)_{\sim r} \\ &= x_h(t_n)_{\sim r} - x(t_n)_{\sim r} + \sum_{j=j_0}^{q-1} w_{nj}(h) r(t_j). \end{aligned} \quad (34)$$

Since  $q + \gamma \geq p$ , we can apply the inequalities (26) with the function  $r$  given by (33) instead of  $f$  thus arriving at the estimate

$$|x_h(t_n)_{\sim r} - x(t_n)_{\sim r}| \leq Cn^{\mu-1+q+\gamma-p}h^{\mu-1+q+\gamma}. \quad (35)$$

For  $k = 0, 1, \dots, q-1$  we have  $k + \gamma < p$  and the inequalities (26) give us the estimate

$$|x_h(t_n)_{\sim t^{k+\gamma-1}} - x(t_n)_{\sim t^{k+\gamma-1}}| \leq Cn^{\mu-1}h^{\mu-1+k+\gamma}.$$

Combining this last estimate with (29) we get

$$|w_{nj}(h)| \leq Cn^{\mu-1}h^{\mu}, \quad n = 1, \dots, N(h), \quad j = j_0, \dots, q-1. \quad (36)$$

It follows from (33) that

$$|r(t_j)| \leq Ch^{q+\gamma-1}, \quad j = j_0, \dots, q-1. \quad (37)$$

The estimates (34)–(37) lead to the estimate (30). ■

**Remark 5** *It was shown in [21] that the analytical solution of the problem (1) with zero initial conditions and with the power function  $f(t) = t^\alpha$ ,  $\alpha > -1$  in the right-hand side is given by*

$$x(t)_{\sim t^\alpha} = \Gamma(\alpha + 1)E(\mu + \alpha + 1; t).$$

*Then, to evaluate the numbers  $x(t_n)_{\sim t^{k+\gamma-1}}$ ,  $k = 0, \dots, q-1$ ,  $n = 1, \dots, N(h)$ , needed to build the system (29), the relation*

$$x(t_n)_{\sim t^{k+\gamma-1}} = \Gamma(k + \gamma)E(k + \mu + \gamma; t_n), \quad k = 0, \dots, q-1, \quad n = 1, \dots, N(h),$$

*and the approximation (13) to evaluate the Mittag-Leffler type functions can be used.*

**Remark 6** *The estimate of type (30) for the approximation of the convolution integral*

$$(g * f)(t) = \int_0^t g(\tau)f(t - \tau) d\tau$$

*by convolution quadrature formula with the weights determined with the help of the Laplace transform of the function  $g$  (in our case it is the function  $F$  given by (9) with  $\beta = \mu$ ) was proved in the case  $\gamma = 1$  ( $q = p - 1$ ) by Lubich [19]. He also showed that for  $t$  bounded away from 0, i.e. for  $t_n \in [t_0, T]$  with a fixed  $t_0 > 0$ , the quadrature weights  $w_{nj}(h)$  in (28) can be chosen as follows:*

$$w_{nj}(h) = c_j\omega_{n-j}(h), \quad j = 0, \dots, p-2, \quad (38)$$

*where the coefficients  $\omega_j(h)$  are given by (14) and  $c_j$  are the correction weights of the  $p$ -th order Newton-Gregory formula (end-point correction of the trapezoidal rule). For example,*

- $c_0 = -1/2$  for  $p = 2$ ,
- $c_0 = -7/12$ ,  $c_1 = 1/12$  for  $p = 3$ ,
- $c_0 = -5/8$ ,  $c_1 = 1/6$ ,  $c_2 = -1/24$  for  $p = 4$ .

In the general case of the problem (1) the numerical scheme is constructed by adding the approximation  $x_h$  given by (17) (the case of homogeneous problem with non-zero initial conditions) to the approximation  $\tilde{x}_{h \sim f}$  given by (28) (the case of inhomogeneous problem with zero initial conditions). Theorems 3 and 5 and estimates (22), (30) lead to the convergence of order  $p$  of this approximation. We recall that  $p$  is the order of the underlying multistep method for the convolution quadrature method that we are using. We proved this result if the right-hand side of the equation (1), i.e., the function  $f$ , is represented in the form  $f(t) = t^{\gamma-1} \tilde{f}(t)$  with  $\tilde{f} \in C^p[0, T]$ ,  $p \geq 1$  and  $\gamma > 0$  if  $\mu \in \mathbb{N}$  and  $\gamma \geq m - \mu$  if  $\mu \notin \mathbb{N}$ .

Evidently, our algorithm can be applied to the initial value problem (1) with the right-hand side of the form

$$f(t) = \sum_{j=1}^k f_j(t), \quad f_j(t) = t^{\gamma_j-1} \tilde{f}_j(t).$$

The numerical scheme is given in this case by

$$\hat{x}_h(t_n)_{\sim f} := x_h(t_n) + \sum_{j=1}^k \tilde{x}_h(t_n)_{\sim f_j}, \quad (39)$$

where  $x_h(t_n)$  and  $\tilde{x}_h(t_n)_{\sim f}$  are defined by (17) and (28), respectively. For this numerical scheme, we arrive at the following result:

**Theorem 6** *Let  $\hat{x}_h$  be the approximation (39) of the solution  $x$  of the problem (1) with the right-hand side in the form  $f(t) = \sum_{j=1}^k f_j(t)$ ,  $f_j(t) = t^{\gamma_j-1} \tilde{f}_j(t)$ , with  $\tilde{f}_j \in C^p[0, T]$ ,  $p \geq 1$  and  $\gamma_j > 0$  if  $\mu \in \mathbb{N}$  and  $\gamma_j \geq m - \mu$  if  $\mu \notin \mathbb{N}$ . Then*

$$|\hat{x}_h(T) - x(T)| = O(h^p), \quad h \rightarrow 0, \quad (40)$$

where a linear multistep method used to generate the approximation (39) is consistent of order  $p$ .

## 4 Related work

General numerical methods explicitly designed for this type of problem are, to the best of our knowledge, almost unknown, the sole exception being the Ph. D. thesis of Nkamnang [23]. At first sight his method looks very similar to ours because he uses many of the ingredients that we do, but these ingredients are combined in a significantly different way, which in the end results in a

different type of algorithm. Specifically he rewrites the fractional differential equation in the form of an equivalent Volterra equation. Then he discretizes this equation and analyzes this discretization with respect to convergence order. Our approach on the other hand is based on the fact that analytical expressions for the exact solutions and their Laplace transforms are known. We use Lubich's numerical scheme to invert these Laplace transforms and find an approximate solution in this way. Thus we do not discretize the originally given initial value problem.

We note in particular that the usual approach for differential equations of (higher) integer order, namely the transformation to a system of equations of first order, is not applicable to our problem: It only works if the orders of the various differential operators involved have a common divisor, and this concept is clearly not useful if arbitrary real numbers are allowed as orders. In certain special cases such as, e.g., the Bagley-Torvik equation (3) this principle is applicable; this has been investigated in [6]. In the present paper however we wanted to discuss the general setting without any number-theoretic assumptions on the orders  $\mu, \mu_1, \mu_2, \dots, \mu_\nu$ .

A number of methods for the numerical solution of one-term differential equations of fractional order have been proposed; we specifically mention those of [3] and [8] that allow a generalization to multi-term equations in the following way.

The method of [3] has been designed for the solution of the equation

$$(D_*^\mu x)(t) - \lambda_1 x(t) = f(t),$$

which is the special case  $\nu = 1$ ,  $\mu_1 = 0$ , of our general problem (1). The fundamental idea is a discretization of the differential operator, based on the representation

$$(D_*^\mu x)(t) = \frac{1}{\Gamma(-\mu)} \int_0^t \frac{x(\tau) - T_{m-1}(\tau)}{(t-\tau)^{\mu+1}} d\tau, \quad (41)$$

where  $T_{m-1}$  is the Taylor polynomial of degree  $m-1$  for the function  $x$  centered at 0, with the quantity  $m$  being defined by the relations

$$m \in \mathbb{N}, \quad m-1 < \mu < m.$$

It follows from a result of Elliott [12] that the representations (41) and (2) are identical if the function  $x$  is sufficiently smooth, which is true in the situation discussed here. The strongly singular integral in (41) must be interpreted in Hadamard's finite-part sense [16]. This integral is then approximated using a product trapezoidal quadrature formula with respect to the weight function  $(t-\cdot)^{-\mu-1}$ , cf. [4]. As a result, an algebraic equation for the unknown values  $x_j \approx x(t_j)$ ,  $j = 0, 1, \dots, N$ , is obtained. Since the method is essentially based on a direct discretization of the differential operator using an analogue of backward differences, we may interpret the scheme as a generalization of the classical backward differentiation formula (BDF approach). In [3] the method was introduced and analyzed for  $0 < \mu < 1$ , but the extension to  $1 < \mu < 2$  presents

no major difficulties. From [3] we know the error bound  $x_j - x(t_j) = O(h^{2-\mu})$ , where  $h$  is the mesh size of the underlying uniform discretization, if the solution is sufficiently smooth. It is easily seen that this still holds under our more general assumption  $0 < \mu < 2$ . As a particular consequence of this error bound, we derive that convergence can only be obtained whenever  $\mu < 2$ . For our general multi-term equation (1) we can discretize all the differential operators in this way and combine these approximations. Obviously, this yields an algorithm convergent as  $O(h^{2-\mu})$ , and we can only use it if  $\mu_1 < \mu_2 < \dots < \mu_\nu < \mu < 2$ . This restriction, for example, prohibits the application to the Bagley-Torvik equation (3).

The method of [8] (investigated further in [9]) does not show this weakness. It has been constructed with the general nonlinear problem

$$(D_*^\mu x)(t) = f(t, x(t)) \quad (42)$$

in mind. Under suitable smoothness assumptions (that may be difficult to verify) it converges as  $O(h^2)$ , independent of  $\mu$ , and therefore there is no need to impose a restriction on  $\mu$ . The method is based on the application of the Riemann-Liouville integral operator of order  $\mu$  (see [24, 26]),

$$(J^\mu x)(t) = \frac{1}{\Gamma(\mu)} \int_0^t x(\tau)(t-\tau)^{\mu-1} d\tau,$$

to (42), thus obtaining the equation

$$x(t) = \sum_{k=0}^{m-1} \frac{t^k}{k!} x^{(k)}(0) + \frac{1}{\Gamma(\mu)} \int_0^t f(\tau, x(\tau))(t-\tau)^{\mu-1} d\tau \quad (43)$$

which is a (possibly weakly singular) Volterra integral equation. Here again  $m$  is the integer defined by  $m-1 < \mu < m$ . Note that the quantities  $x^{(k)}(0)$  appearing on the right-hand side of the Volterra equation (43) are given because of the prescribed initial conditions. The discretization of (43) is then based on the classical PECE (predict, evaluate, correct, evaluate) principle well known for first-order ordinary differential equations: We start with an approximation of the integral on the right-hand side of (43) using the product rectangle method, thus obtaining a prediction for the unknown solution, and then we approximate the integral once more, using the product trapezoidal method, thus obtaining the corrector, which is then accepted as the final approximation. We may generalize this concept to (1) as well: Applying the operator  $J^\mu$  to (1), we obtain the Volterra equation

$$x(t) = \sum_{k=0}^{m-1} \frac{t^k}{k!} x^{(k)}(0) + \sum_{i=1}^{\nu} \lambda_i (J^{\mu-\mu_i} [x - T_{m_i-1}])(t) + (J^\mu f)(t)$$

(cf., e.g., [23, eq. (3.186)]). For the interpretation of this relation we recall that  $m_i$  is the integer satisfying  $m_i - 1 < \mu_i \leq m_i$ , and that  $T_{m_i-1}$  is the Taylor



polynomial of degree  $m_i - 1$  for the function  $x$  centered at 0. Using the explicit representation of the operators  $J^\mu$  and  $J^{\mu-\mu_i}$ , we may rewrite this equation as

$$x(t) = g(t) + \sum_{i=1}^{\nu} \frac{\lambda_i}{\Gamma(\mu - \mu_i)} \int_0^t x(\tau)(t - \tau)^{\mu - \mu_i - 1} d\tau \quad (44)$$

where

$$\begin{aligned} g(t) = & \sum_{k=0}^{m-1} \frac{x^{(k)}(0)}{k!} t^k - \sum_{i=1}^{\nu} \lambda_i \sum_{k=0}^{m_i-1} \frac{x^{(k)}(0)}{\Gamma(k+1+\mu-\mu_i)} t^{k+\mu-\mu_i} \\ & + \frac{1}{\Gamma(\mu)} \int_0^t f(\tau)(t - \tau)^{\mu-1} d\tau \end{aligned}$$

is a known function. Then we can solve (44) numerically by the approach described above.

In the next section, we shall compare the results of these two approaches with numerical results obtained by the new method introduced in the present paper.

We note that some other approaches for numerical solution of the general non-linear differential equations of fractional order have been described in the literature. These approaches do not use the advantages of the special form of the problem (1) (that however covers most cases of modelling of processes with differential equations of fractional order in applications) and thus lead in general to lower approximation orders. For completeness we mention some recent publications concerning these general approaches (see also references therein). In the paper [18] a method of decomposition of the fractional differential equations (both linear and non-linear ones) into systems of integro-differential equations of a special kind was suggested. The systems composed of one ordinary differential equation of integer order and a number of left inverse equations of the Abel integral operator were then used to numerical solution of the problem under consideration. In the recent paper [11] a number of numerical examples of linear and non-linear differential equations of fractional order solved by different methods is presented. For an extensive and up-to-date review of numerical methods for problems described by fractional-order derivatives, integrals, and differential equations we refer the interested reader to the recent contribution by Diethelm, Ford, Freed and Luchko [7].

## 5 Numerical examples

In this section we give some examples of numerical solution of the problem (1) by means of the approximation described in Sections 2 and 3. To better analyze the numerical results we have chosen those problems which have analytical solutions of a simple type. In all cases we evaluate the experimental order of convergence (EOC) by using the formula

$$p_e(t_n) = \log_2 \left( \frac{|x(t_n) - \hat{x}_h(t_n)|}{|x(t_n) - \hat{x}_{h/2}(t_{2n})|} \right), \quad t_n \in G_h(T),$$

where the analytical solution  $x$  of the problem (1) is evaluated independently. The solutions in form of combinations of elementary and known special functions used in our examples were obtained by the “Simplify” procedure of MATHEMATICA applied to the solutions in form of the Mittag-Leffler type functions.

The results are compared to results obtained by the (generalized versions of the) methods of [3] and [8] as discussed in the previous section.

In all examples, except for the last one, we looked at the equation on the interval  $[0, 1]$ . In the following tables we give the approximation errors at  $t = 1$  and the experimental convergence orders for the various numerical methods discussed above. The notation of the tables is as follows. By  $h$  we denote the step size of the algorithm, the column headed by “PECE” gives the results (error and experimental order of convergence, EOC) for the PECE method generalizing the approach of Diethelm and Freed [8, 9] (cf. §4), and in the “BDF” column we have the corresponding results for the method of Diethelm [3], cf. also §4. Moreover, the results obtained by the new convolution quadrature method (CQM) proposed in this paper with different values of  $p$  (the consistency order of the multistep method underlying for the convolution quadrature method) are given in the third column.

We begin our exposition with some homogeneous equations; inhomogeneous problems will be considered subsequently.

**Example 1** Our first example is the equation  $D_*^{1/4}y(t) + y(t) = 0$ ,  $y(0) = 1$ . The exact solution is

$$y(t) = e^t(\operatorname{erf} \sqrt{t} - 1) + e^t \left( \frac{\Gamma(1/4, t)}{\Gamma(1/4)} + \frac{\Gamma(3/4, t)}{\Gamma(3/4)} \right)$$

where  $\Gamma(x, t)$  is the incomplete Gamma function.

Here we found the following results.

$h$	PECE		BDF		CQM, $p = 2$	
	error	EOC	error	EOC	error	EOC
0.1	−0.002124		−0.003417		−0.000272	
0.05	−0.000671	1.66	−0.001658	1.04	−0.0000622	2.19
0.025	−0.000229	1.55	−0.000815	1.02	−0.0000151	2.07
0.0125	−0.000081	1.49	−0.000404	1.01	−0.0000037	2.04

**Example 2** Next we look at  $D_*^{1/2}y(t) - y(t) = 0$  with the initial condition chosen as  $y(0) = 1$  such that the solution is

$$y(t) = e^t(\operatorname{erf} \sqrt{t} + 1).$$

Here we found the following results.

$h$	PECE		BDF		CQM, $p = 2$	
	error	EOC	error	EOC	error	EOC
0.1	0.1910		0.1060		−0.0665	
0.05	0.0732	1.38	0.0751	0.50	−0.0171	1.95
0.025	0.0270	1.44	0.0461	0.70	−0.00434	1.97
0.0125	0.0098	1.46	0.0263	0.81	−0.00109	1.98

**Example 3** Our third example is an equation where  $\mu > 1$ , namely  $D_*^{3/2}y(t) - y(t) = 0$  with the initial conditions  $y(0) = 1$  and  $y'(0) = -1$ . The analytical solution can be written through the special functions of the hypergeometric type in the form

$$y(t) = \frac{2}{15\sqrt{\pi}} \left( {}_{10}F_3(1; 5/6, 7/6, 3/2; t^3/27)t^{3/2} - 4 {}_1F_3(1; 7/6, 3/2, 11/6; t^3/27)t^{5/2} + 5\sqrt{\pi}e^{-t/2}[\cos(\sqrt{3}t/2) + \cos(\frac{1}{6}(3\sqrt{3}t + 2\pi))] \right).$$

Here we found the following results.

$h$	PECE		BDF		CQM, $p = 2$	
	error	EOC	error	EOC	error	EOC
0.1	-0.000968		-0.3381		-0.00121	
0.05	-0.000236	1.48	-0.2198	0.62	-0.000283	2.14
0.025	-0.000058	2.03	-0.1482	0.57	-0.000689	2.05
0.0125	-0.000014	2.02	-0.1018	0.54	-0.000017	2.02

**Example 4** The first inhomogeneous equation we considered was  $D_*^{1/4}y(t) - y(t) = t^2 + 2t^{7/4}/\Gamma(11/4)$  with the initial condition being  $y(0) = 0$  such that the solution is

$$y(t) = -t^2 - \frac{16t^{3/2}}{3\sqrt{\pi}} - 4t - \frac{8\sqrt{t}}{\sqrt{\pi}} + 12e^t + 4e^t \operatorname{erf} \sqrt{t} - \frac{4}{\Gamma(9/4)}e^t\Gamma(9/4, t) - \frac{4}{\Gamma(11/4)}e^t\Gamma(11/4, t) - 4.$$

Here we found the following results.

$h$	PECE		BDF		CQM, $p = 2$	
	error	EOC	error	EOC	error	EOC
0.1	1.0998		-0.25577		-0.146	
0.05	0.5444	1.01	-0.08327	1.62	-0.0381	1.91
0.025	0.2484	1.13	-0.02669	1.64	-0.00974	1.95
0.0125	0.1087	1.19	-0.00843	1.66	-0.00247	1.97

**Example 5** A similar example is  $D_*^{1/2}y(t) + y(t) = t^2 + 2t^{3/2}/\Gamma(5/2)$  with the initial condition  $y(0) = 0$  such that the solution is  $y(t) = t^2$ .

Here we found the following results.

$h$	PECE		BDF		CQM, $p = 2$	
	error	EOC	error	EOC	error	EOC
0.1	0.01671		-0.007724		-0.00145	
0.05	0.00523	1.68	-0.002815	1.45	-0.000359	2.016
0.025	0.00168	1.63	-0.001015	1.47	-0.0000896	2.006
0.0125	0.00056	1.60	-0.000364	1.48	-0.0000223	2.002

**Example 6** Now we move again to equations with  $\mu > 1$ , specifically  $D_*^{3/2}y(t) + y(t) = t^2 + 2t^{1/2}/\Gamma(3/2)$  with the initial conditions  $y(0) = 0$  and  $y'(0) = 0$  such that the solution is again  $y(t) = t^2$ .

Here we found the following results.

$h$	PECE		BDF		CQM, $p = 2$	
	error	EOC	error	EOC	error	EOC
0.1	0.001249		-0.3649		0.00592	
0.05	0.000294	2.08	-0.2508	0.54	0.00254	1.16
0.025	0.000071	2.06	-0.1725	0.54	0.000958	1.33
0.0125	0.000017	2.05	-0.1191	0.53	0.000343	1.41

**Example 7** A similar example (with a different sign for  $\lambda_1$  though) is the differential equation  $D_*^{3/2}y(t) - y(t) = t^2 + 2t^{1/2}/\Gamma(3/2)$  with the initial condition  $y(0) = y'(0) = 0$ . The solution is

$$y(t) = \frac{64}{105\sqrt{\pi}} {}_1F_3(1; 3/2, 11/6, 13/6; t^3/27)t^{7/2} - t^2 + \frac{4}{3}e^t - \frac{8}{3}e^{-t/2} \cos\left(\frac{1}{6}(2\pi - 3\sqrt{3}t)\right).$$

Here we found the following results.

$h$	PECE		BDF		CQM, $p = 1$	
	error	EOC	error	EOC	error	EOC
0.1	-0.001871		-1.2815		-0.332	
0.05	-0.000494	1.92	-0.7438	0.78	-0.158	1.05
0.025	-0.000128	1.94	-0.4639	0.68	-0.0777	1.017
0.0125	-0.000033	1.96	-0.3018	0.62	-0.0387	1.003

**Example 8** Finally we consider the Bagley-Torvik equation  $D^2y(t) + D_*^{3/2}y(t) + y(t) = t + 1$  with the initial conditions  $y(0) = y'(0) = 1$ . In this case the exact solution is  $y(t) = t + 1$ . In contrast to the previous examples we now look at the errors at  $t = 5$ . This is the example considered in [6].

Here we found the following results.

$h$	PECE		CQM, $p = 2$	
	error	EOC	error	EOC
0.5	0.3831		0.00741	
0.25	0.0904	2.08	0.00630	
0.125	0.0265	1.77	0.00196	1.60
0.0625	0.0084	1.65	0.00056	1.74
0.03125	0.0028	1.59	0.00016	1.76

Summarizing these numerical results, we can say that the PECE method always gives a reasonable convergence behaviour. Specifically the convergence seems to be  $O(h^2)$  in Examples 3, 6, and 7 where only differential operators of order  $3/2$  were contained. Following [8, 9], this is the best we may expect. In

Examples 1, 2, 4 and 5 we seem to have convergence as  $O(h^{1+\mu})$ . In Example 8 we have  $O(h^{3/2})$ . This is the same rate of convergence as observed in [6] where a different approach has been taken to generalize the PECE scheme to the Bagley-Torvik equation. Note however that, even though the convergence orders seem to be identical in these two cases, the implied constants are smaller in the approach of [6].

For the BDF method, we find results confirming the theoretical bounds derived in [3], i.e. a convergence order of  $O(h^{2-\mu})$ , in Examples 3, 4, 5, 6, and 7. In Example 8, as mentioned in the introduction, the scheme cannot be applied at all. In the first two examples, the convergence is only  $O(h)$ . This is due to the fact that the solution is not sufficiently smooth (not differentiable at the origin), and therefore it cannot be approximated by the full order of the method (i.e. the smoothness assumptions required in the derivation of the error bound are violated).

We thus find that the PECE approach is superior to the BDF method in the cases where  $\mu > 1/2$  (Examples 3, 6, 7, and 8) and in those cases when the solution is not very smooth (Examples 1 and 2). Example 4 is a case with  $\mu < 1/2$  and a smooth solution. Here the BDF method works better than the PECE algorithm. In Example 5, we have  $\mu = 1/2$  and a smooth solution. In such a case both methods give very similar results.

The CQM method works well for homogeneous equations (Examples 1, 2, and 3) and for those inhomogeneous equations where the right-hand side  $f$  can be represented in the form  $f(t) = t^{\gamma-1}\tilde{f}(t)$  with a sufficiently smooth function  $\tilde{f} \in C^p[0, T]$  and  $\gamma > 0$  (Examples 4, 5, and 8). In Examples 6 and 7 we can represent the function  $f_1(t) = t^{1/2}$ , the “worst” of the terms from the right-hand side of the equation, in the form  $f_1(t) = t^{\gamma-1}t^{p+\varepsilon}$ ,  $\gamma > 0$ ,  $\varepsilon \geq 0$ . It follows from this representation that  $p = 1.5 - \gamma - \varepsilon < 1.5$ . According to Theorem 6 the CQM can give in this case the convergence order  $p$  if we choose the consistent order of the multistep method underlying for the convolution quadrature method to be equal to  $p$ . In Example 7 we take  $p = 1 < 1.5$  and get the EOC tending to 1. In Example 6 we try to use the multistep method with the consistent order  $p = 2$  but the EOC of CQM is restricted by 1.5 as could be expected from our theoretical considerations.

We thus conclude that for homogeneous equations and for inhomogeneous equations with the right-hand side  $f$  in the form  $f(t) = t^{\gamma-1}\tilde{f}(t)$  with a sufficiently smooth function  $\tilde{f}$  and  $\gamma > 0$  the CQM method gives as a rule better approximations than both the PECE and BDF methods; for the equations with special right-hand sides such as in Examples 6 and 7 the preference should be given to the PECE approach.

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# Differential equations of some classes of Special functions via the Factorization method

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## Abstract

Let  $\{P_n(x)\}_{n=0}^{\infty}$  be a sequence of polynomials of degree  $n$ . We define two sequences of differential operators  $\Phi_n$  and  $\Psi_n$  satisfying the following properties

$$\Phi_n(P_n(x)) = P_{n-1}(x),$$

$$\Psi_n(P_n(x)) = P_{n+1}(x).$$

By constructing these two operators for some classes of special functions, we determine their differential equations via the factorization method introduced in [3]. We illustrate our method by including classical orthogonal polynomials,  $d$ -orthogonal polynomials, confluent hypergeometric functions and hypergeometric functions as the applications.

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## 1. Introduction

Let  $\{P_n(x)\}_{n=0}^{\infty}$  be a sequence of polynomials of degree  $n$ . For  $n = 0, 1, 2, \dots$ , we define two sequences of operators  $\Phi_n$  and  $\Psi_n$  satisfying the following properties

$$\Phi_n(P_n(x)) = P_{n-1}(x),$$

$$\Psi_n(P_n(x)) = P_{n+1}(x).$$

$\Phi_n$  and  $\Psi_n$  play the role analogous to that of derivative and multiplicative operators respectively on monomials. The monomiality principle and the associated operational rules were used in [10] to explore new classes of isospectral problems leading to non trivial generalizations of special functions. Most of properties of families of polynomials associated with these two operators can be deduced using operator rules with the  $\Phi_n$  and  $\Psi_n$  operators. The operators we defined in this paper are varying with the degrees of polynomials  $n$ . The iterations of  $\Phi_n$  and  $\Psi_n$  to  $P_n(x)$  give us the following relations:

$$(\Phi_{n+1}\Psi_n)P_n(x) = P_n(x)$$

$$(\Psi_{n-1}\Phi_n)P_n(x) = P_n(x)$$

$$(\Phi_1\Phi_2\cdots\Phi_{n-1}\Phi_n)P_n(x) = P_0(x)$$

$$(\Psi_{n-1}\Psi_{n-2}\cdots\Psi_1\Psi_0)P_0(x) = P_n(x).$$

These operational relations allow us to derive a higher order differential equation satisfied by some special polynomials. The classical factorization method introduced in [3] was used to study the second-order differential equation.

In this paper we construct  $\Phi_n$  and  $\Psi_n$  for some classes of Polynomials satisfying recursion relations and derive the corresponding differential equations. We illustrate the factorization methods by applying the operators to some special functions such as orthogonal polynomials,  $d$ -orthogonal polynomials, confluent hypergeometric functions and hypergeometric functions in Section 3.

## 2. Operators $\Phi_n$ and $\Psi_n$

In this section we consider two classes of polynomials satisfying some recurrence relations. The first recursion relation for the polynomials is a

$(n + 2)$ -term recursion formula. The number of terms of recurrence is varying with the degree of the polynomials. The Bernoulli and Euler polynomials and some of their generalizations are examples of these types of polynomials. The differential equations of these polynomials are found by using the factorization method in [7]. The second kind recursion relation is a  $(d + 2)$ -term recurrence relation with  $d \geq 1$ . The number of the terms of recurrence is fixed and greater than or equal to 3. There are many classes of polynomials with these types of recursion formulas such as orthogonal polynomials and confluent hypergeometric functions. We introduce two sequences of differential operators  $\Phi_n$  and  $\Psi_n$  for each class of the polynomials corresponding the recursion formulas.

The first theorem deals with the polynomials satisfying  $(n + 2)$ -term recursion relation.

**Theorem 2.1** *Let  $\{P_n(x)\}_{n \geq 0}$  be a sequence of polynomials satisfying the following  $(n + 2)$ -term recursion relation*

$$P_{n+1}(x) = (x - \alpha_n)P_n(x) + \sum_{k=1}^n \beta_{n-k}P_{n-k}(x), \quad (2.1)$$

with initial condition  $P_0(x) = 1$ . If there is a differential operator  $\Phi_n$  such that

$$\Phi_n P_n(x) = P_{n-1}(x), \quad (2.2)$$

then the polynomial sequence  $\{P_n(x)\}_{n \geq 0}$  satisfies the following differential equation

$$(\Phi_{n+1}\Psi_n)P_n(x) = P_n(x), \quad (2.3)$$

where

$$\Psi_n = (x - \alpha_n) + \sum_{k=1}^n \beta_{n-k} \prod_{j=1}^k \Phi_{n-k+j}. \quad (2.4)$$

**Proof.** Apply the operator  $\Phi_n$  (2.2) iteratively in (2.1) so that for  $k = 1, 2, \dots, n$ ,

$$P_{n-k}(x) = [\Phi_{n-k+1}\Phi_{n-k+2} \cdots \Phi_n]P_n(x) = \left(\prod_{j=1}^k \Phi_{n-k+j}\right)P_n(x),$$

where  $\prod_{j=1}^k \Phi_{n-k+j}$  is in this order. Then the recursion relation (2.1) can be written as

$$P_{n+1}(x) = (x - \alpha_n)P_n(x) + \sum_{k=1}^n \beta_{n-k} \left( \prod_{j=1}^k \Phi_{n-k+j} \right) P_n(x).$$

That is,

$$P_{n+1}(x) = \Psi_n P_n(x),$$

where  $\Psi_n$  is defined as in (2.4). The equation (2.3) is a simple multiplication of operators  $\Phi_{n+1}$  and  $\Psi_n$ . This completes our proof.  $\blacksquare$

The next theorem deals with the polynomials satisfying  $(d+2)$ -term recursion relation. Since the proof is based on a proper iteration of the operators and is similar to the first theorem, we only state the result and omit the proof.

**Theorem 2.2** *Let  $\{P_n(x)\}_{n \geq 0}$  be a sequence of polynomials satisfying the following  $(d+2)$ -term recursion relation*

$$P_{n+d+1}(x) = (x - \alpha_{n+d})P_{n+d}(x) + \sum_{k=1}^d \beta_{n+d-k} P_{n+d-k}(x), \quad (2.5)$$

*with some proper initial conditions. If there is a differential operator  $\Phi_n$  such that*

$$\Phi_n P_n(x) = P_{n-1}(x), \quad (2.6)$$

*then the polynomial sequence  $\{P_n(x)\}_{n \geq 0}$  satisfies the following differential equation*

$$(\Phi_{n+1} \Psi_n) P_n(x) = P_n(x), \quad (n \geq d) \quad (2.7)$$

*where,*

$$\Psi_n = (x - \alpha_n) + \sum_{k=1}^d \beta_{n-k} \prod_{j=1}^k \Phi_{n-k+j}. \quad (2.8)$$

*If there is an operator  $\Psi_n$  such that*

$$\Psi_n P_n(x) = P_{n+1}(x), \quad (2.9)$$

then the operator  $\Phi_{n+1}$ ,  $\forall n \geq 0$ , is given by

$$\Phi_{n+1} = \frac{1}{\beta_n} \left\{ \prod_{k=1}^d \Psi_{n+k} - \left[ (x - \alpha_{n+d}) \prod_{k=1}^{d-1} \Psi_{n+k} + \sum_{k=1}^{d-1} \beta_{n+d-k} \prod_{j=1}^{d-k-1} \Psi_{n+j} \right] \right\}. \quad (2.10)$$

In the following section we illustrate the factorization method for some special functions. The operators  $\Phi_n$  and  $\Psi_n$  will be given explicitly.

### 3. Applications

In this section, we illustrate our factorization method for various special functions. We'll begin with classical orthogonal polynomials and  $d$ -orthogonal polynomials in the following section. We then use the confluent hypergeometric and hypergeometric functions to further demonstrate our method.

For each special function, we shall provide explicitly the operators of  $\Phi_n$  and  $\Psi_n$ . The differential equations are the easy consequences of the application of operators of  $\Phi_n$  and  $\Psi_n$  to each special function.

#### 3.1 Differential Equation for classical orthogonal polynomials

It is well known that all orthogonal polynomials  $\{P_n\}$  satisfy recursion formulas [6] connecting three consecutive polynomials  $P_{n-1}(x)$ ,  $P_n(x)$ ,  $P_{n+1}(x)$ :

$$\alpha_n P_{n+1}(x) = (x - \beta_n) P_n(x) - \gamma_n P_{n-1}(x),$$

where  $\alpha_n$ ,  $\beta_n$ , and  $\gamma_n$  are constants. Furthermore, all orthogonal polynomials satisfy the differential formula [6]:

$$\sigma(x) P'_n(x) = (A_n x + B_n) P_n(x) + C_n P_{n-1}(x),$$

where  $\sigma(x)$  is a polynomial of degree at most 2,  $A_n$ ,  $B_n$  and  $C_n$  are constants. Using the differential formula, we easily get a differential operator:

$$\Phi_n = \frac{1}{C_n} [\sigma(x) D_x - A_n x - B_n]$$

such that

$$\Phi_n P_n(x) = P_{n-1}.$$

Using the recursion formula for  $P_n(x)$  together with the operator  $\Phi_n$ , we get the operator:

$$\Psi_n = \frac{1}{\alpha_n} [(x - \beta_n) - \gamma_n \Phi_n]$$

such that

$$\Psi_n P_n(x) = P_{n+1}(x).$$

The differential equations for  $P_n(x)$  are the consequences of the following relation

$$\Phi_{n+1} \Psi_n P_n(x) = P_n(x).$$

We now provide a summary of operators  $\Phi_n$  and  $\Psi_n$  for all classical orthogonal polynomials. Since all the calculation are similar, we omit the details.

#### $\Phi_n$ and $\Psi_n$ for classical orthogonal polynomials

$p_n(x)$	$\Phi_n$	$\Psi_n$
$P_n(x)$	$x + \frac{1-x^2}{n} D_x$	$x - \frac{1-x^2}{n+1} D_x$
$T_n(x)$	$x + \frac{1-x^2}{n} D_x$	$x - \frac{1-x^2}{n} D_x$
$U_n(x)$	$\frac{n}{n+1}x + \frac{1-x^2}{n+1} D_x$	$\frac{n+2}{n+1}x - \frac{1-x^2}{n+1} D_x$
$H_n(x)$	$\frac{1}{2n} D_x$	$2x - D_x$
$L_n^{(\alpha)}(x)$	$\frac{n}{n+\alpha} - \frac{x}{n+\alpha} D_x$	$\frac{n+\alpha+1-x}{n+1} + \frac{x}{n+1} D_x$
$C_n^{(\lambda)}(x)$	$\frac{1}{2\lambda+n-1}(nx + (1-x^2)D_x)$	$\frac{(n+2\alpha)x - (1-x^2)D_x}{n+1}$

For the Jacobi polynomials  $J_n^{(\alpha, \beta)}(x)$ , the operators are given by

$$\Phi_n = \frac{n}{n+\alpha} - \frac{(1-x)n(2n+\alpha+\beta)}{2(n+\beta)(n+\alpha)} - \frac{(1-x^2)(2n+\alpha+\beta)}{2(n+\beta)(n+\alpha)} D_x,$$

$$\begin{aligned}\Psi_n &= \frac{(\alpha + \beta + 2n - 1)\{\alpha^2 - \beta^2 + x(\alpha + \beta + 2n)(\alpha + \beta + 2n - 2)\}}{2n(\alpha + \beta + n)(\alpha + \beta + 2n - 2)} \\ &- \frac{2(\alpha + n - 1)(\beta + n - 1)(\alpha + \beta + 2n)\Phi_{n-1}}{2n(\alpha + \beta + n)(\alpha + \beta + 2n - 2)}.\end{aligned}$$

■

### 3.2 Differential Equation for some $d$ -Orthogonal Polynomials

The polynomial sequence  $\{P_n(x)\}_{n \geq 0}$  is called a  $d$ -orthogonal polynomial sequence (d-OPS) with respect to the  $d$ -dimensional functional  $\mathbf{U} = (u_0, u_1, \dots, u_{d-1})$  if it fulfills [4], [5], [8]

$$\langle u_k, P_m P_n \rangle = 0, \quad m \geq dn + k + 1, \quad n \geq 0,$$

$$\langle u_k, P_n P_{dn+k} \rangle \neq 0, \quad n \geq 0,$$

for each integer  $k$  with  $k = 0, 1, \dots, d-1$ . Note that, when  $d = 1$ , we meet the ordinary regular orthogonality. In this sense,  $\{P_n\}_{n \geq 0}$  is an orthogonal polynomial sequence (OPS). The remarkable characterization of the  $d$ -OPS is that they satisfy a  $(d+1)$ -order recurrence relation [5] which we write in form

$$P_{m+d+1}(x) = (x - \beta_{m+d})P_{m+d}(x) - \sum_{k=0}^{d-1} \gamma_{m+d-k}^{d-1-k} P_{m+d-k-1}(x), \quad m \geq 0,$$

with initial conditions

$$P_0(x) = 1, \quad P_1(x) = x - \beta_0,$$

$$P_n(x) = (x - \beta_{n-1})P_{n-1}(x) - \sum_{k=0}^{n-2} \gamma_{n-1-k}^{d-1-k} P_{n-2-k}(x), \quad 2 \leq n \leq d,$$

and the regularity conditions

$$\gamma_{n+1}^0 \neq 0, \quad n \geq 0.$$

When  $d = 1$ , the above recurrence is reducible to the well-known second-order recurrence relation

$$P_{n+2}(x) = (x - \beta_{n+1})P_{n+1}(x) - \gamma_{n+1}P_n(x), \quad n \geq 0;$$

with initial conditions  $P_0(x) = 1$ ,  $P_1(x) = x - \beta_0$ .

We now determine the operators  $\Phi_n$  and  $\Psi_n$  for a set of 2-orthogonal polynomials  $\{B_n^{\alpha,\beta}(x)\}_{n \geq 0}$  introduced in [8]. For the simplicity of notation, we also use the  $\{B_n^{\alpha,\beta}(x)\}_{n \geq 0}$  for its monic version. The following relations are given in [8],

$$D_x B_{n+1}^{\alpha,\beta}(x) = (n+1)B_n^{\alpha+1,\beta+1}(x),$$

$$x B_n^{\alpha+1,\beta+1}(x) = B_{n+1}^{\alpha,\beta}(x) + (n+\alpha+1)(n+\beta+1)B_n^{\alpha,\beta},$$

$$B_{n+3}^{\alpha,\beta}(x) = (x - \beta_{n+2})B_{n+2}^{\alpha,\beta}(x) - \alpha_{n+2}B_{n+1}^{\alpha,\beta} - \gamma_{n+1}B_n^{\alpha,\beta}(x).$$

Using the first two relations, we have

$$(xD_x - (n+1))B_{n+1}^{\alpha,\beta}(x) = (n+1)(n+\alpha+1)(n+\beta+1)B_n^{\alpha,\beta}(x).$$

This gives the differential operator of first order:

$$\Phi_{n+1} = \frac{xD_x - (n+1)}{(n+1)(n+\alpha+1)(n+\beta+1)}.$$

Next we use the operator  $\Phi_n$  together with the 4-term recurrence relation of  $B_n^{\alpha,\beta}(x)$  to get the following relation:

$$B_{n+3}^{\alpha,\beta} = [(x - \beta_{n+2}) - (\alpha_{n+2} + \gamma_n \Phi_{n+1})\Phi_{n+2}]B_{n+2}^{\alpha,\beta}(x).$$

This leads to a second-order differential operator:

$$\Psi_{n+2} = [(x - \beta_{n+2}) - (\alpha_{n+2} + \gamma_n \Phi_{n+1})\Phi_{n+2}]$$

such that

$$\Psi_{n+2}B_{n+2}^{\alpha,\beta} = B_{n+3}^{\alpha,\beta}.$$

By applying both operators  $\Phi_{n+1}$  and  $\Psi_n$  consecutively to  $B_n^{\alpha,\beta}(x)$ , we have

$$(\Phi_{n+1}\Psi_n)B_n^{\alpha,\beta}(x) = B_n^{\alpha,\beta}(x).$$



This can be reduced to a 3rd order differential equation for  $B_n^{\alpha,\beta}(x)$ ,

$$x^2 y''' + (3 + \alpha + \beta)xy'' + [(1 + \alpha)(1 + \beta) - x]y' + ny = 0.$$

### 3.3 Differential Equation for confluent hypergeometric functions

The confluent hypergeometric function [6] can be defined by

$$\Phi(a; c; x) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} \frac{x^n}{n!} \quad (3.3.1)$$

The following relations [6] are used to find the operators  $\Phi_a$  and  $\Psi_a$ . We remark here that the operators  $\Phi_a$  and  $\Psi_a$  are associated with the parameter  $a$  instead of degree of polynomials. The parameter  $a$  is a real number. The first relation is the recurrence relation in terms of  $a$ :

$$(c - a)\Phi(a - 1; c; x) + (2a - c + x)\Phi(a; c; x) = a\Phi(a + 1; c; x). \quad (3.3.2)$$

The second relation is a differential identity:

$$\frac{d}{dx} \Phi(a; c; x) = \frac{a}{c} \Phi(a + 1; c + 1; x). \quad (3.3.3)$$

Using these relations one can find that

- If  $a > 0$ , then  $\Phi_a = \frac{x}{c-a} D_x + \frac{c-a-x}{c-a}$ , and  $\Psi_a = \frac{x}{a} D_x + 1$ .
- If  $a < 0$ , then  $\Phi_a = \frac{x}{a} D_x + 1$ , and  $\Psi_a = \frac{x}{c-a} D_x + \frac{c-a-x}{c-a}$ .
- If  $a = 0$ , it is a critical value, and  $\Phi(0; c; x) = 1$ .

We note that if  $a = c$ , then  $\Phi(a; a; x) = e^x$ , which is a limiting case of the confluent hypergeometric function. The product of  $(\Psi_{a-1}\Phi_a)\Phi(a; c; x) = \Phi(a; c; x)$  or  $(\Phi_{a+1}\Psi_a)\Phi(a; c; x) = \Phi(a; c; x)$  leads to the well-known differential equation

$$xy'' + (c - x)y' - ay = 0$$

satisfied by the confluent hypergeometric function  $\Phi(a; c; x)$ .

### 3.4 Differential Equation for hypergeometric functions

The hypergeometric function [6] can be defined by

$$F(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!} \quad (3.4.1)$$

Starting from the recurrence relation

$$F(a+1, b; c; x) = \frac{2a-c+(b-a)x}{a(1-x)} F(a, b; c; x) - \frac{a-c}{a(1-x)} F(a-1, b; c; x),$$

and the differential identity

$$D_x F(a, b; c; x) = \frac{a}{x} F(a+1, b; c; x) - \frac{a}{x} F(a, b; c; x),$$

it is easy to derive the following expressions for the operators  $\Phi_a$  and  $\Psi_a$ .

- If  $a > 0$ , then

$$\Phi_a = 1 + \frac{x}{a} D_x,$$

and

$$\Psi_a = 1 + \frac{bx}{a-c} - \frac{x(1-x)}{a-c} D_x$$

- If  $a < 0$ , then  $\Phi_a$  and  $\Psi_a$  must be interchanged.

The product of  $(\Psi_{a-1}\Phi_a)F(a, b; c; x) = F(a, b; c; x)$  or  $(\Phi_{a+1}\Psi_a)F(a, b; c; x) = F(a, b; c; x)$  leads to the well-known differential equation

$$x(1-x)y'' + [c - (a+b+1)x]y' - aby = 0$$

satisfied by the hypergeometric function  $F(a, b; c; x)$ .

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# On the existence and non-uniqueness of solutions of the Modified Falkner-Skan equation

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## Abstract

We discuss the existence and non-uniqueness of solutions of the modified Falkner-Skan equation which is governed by the two parameters  $\lambda$  and  $\beta$ . The solution is studied over the range of  $|\beta| \leq 1$  and for various values of  $\lambda$ . It is shown (mainly numerically) that for each  $\beta$  in this range there exists a critical value  $\lambda_c(\beta)$  such that two solutions exist for  $0 < \lambda < \lambda_c(\beta)$ , and one solution exists for  $\lambda = \lambda_c(\beta)$ , and no solution exists for  $\lambda > \lambda_c(\beta)$ . When  $\beta = -1$ , it is shown that two solutions exist only if  $\lambda < -1$ .

## 1 Introduction:

The standard Falkner-Skan equation is given by

$$f'''(\eta) + f(\eta)f''(\eta) + \beta(1 - f'^2(\eta)) = 0 \quad (1)$$

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with initial and boundary conditions

$$f(0) = 0, \quad f'(0) = -\lambda, \quad \text{and} \quad f'(\infty) = 1 \quad (2)$$

This equation is associated with the boundary layer flow over a wedge with included angle  $\pi\beta$  and moving with a constant velocity  $\lambda$ . The function  $f(\eta)$  is the non-dimensional stream function and  $\eta$  is the similarity ordinate. The derivation of this equation from the classical Navier-Stokes equations can be found in many books like Schlichting [9].

Over the last few decades this equation has been studied intensively by many authors. When  $\lambda = 0$  the problem is reduced to the homogeneous case. The existence of a solution in this case, when  $0 \leq \beta \leq 1$ , was first proved by Weyl [11] and uniqueness of the solution subject to the condition

$$f'(\eta) > 0 \quad (3)$$

for  $\eta > 0$  was then established by Iglich (cf. [5] and references therein). With the Condition (3) is dropped, the case was studied by Craven and Peltier [4]. They have extended the early result of Coppel [6] and proved that a solution exists and is unique for  $0 \leq \beta \leq 1$ .

The Case  $\beta > 1$  was also studied by Craven and Peltier[5]. It was shown that a solution that satisfies Equation (1) and the boundary condition (2) exists but this solution does not satisfy the Condition (3). Their numerical results showed that reverse flow solution ( $f'(\eta) < 0$ ) for fixed values of  $\beta > 1$  exists and accordingly they conjectured that given a natural number  $N$ , there exists a solution  $f_N$  such that  $f'_N < 0$  on  $N$  disjoint intervals. These reverse flow solutions thus obtained all differ from those obtained for  $\beta < 0$  in two significant aspects:

1. For  $t > 0$ ,  $f''(t) > 0$
2. Non of these solutions exhibits an overshoot.

The case  $\beta < 0$ , was studied by Hasting [7]. It has been shown that if  $\beta$  is very small then there is exactly one solution such that  $f''(0) < 0$ , and  $-1 < f < 1$  on  $[0, \infty)$ . Furthermore as  $t \rightarrow \infty$ ,  $f'(t) \rightarrow 1$  exponentially. The special case  $\beta = -1$  with the condition  $f(0) = \gamma \neq 0$ , was studied by Yang and Chen [11], where an analytical solution was obtained for two different cases: The first case is when  $\gamma \geq \sqrt{2}$  and  $f''(0) = \sqrt{\gamma^2 - 2}$ , the solution then

is given in terms of an exponential and an error function, and the second case is when  $0 \leq \gamma \leq \sqrt{2}$  and  $f''(0) = 0$ . The solution in this case is given in terms of confluent hypergeometric functions.

The case  $\lambda \neq 0$  was studied recently by Riley and Weidman [8]. They employed numerical methods to study the existence and non-uniqueness of solution for  $|\beta| \leq 1$  over a range of positive and negative values of  $\lambda$ . Their results indicate that for  $-1 \leq \beta \leq 0$ , two solutions exist for  $\lambda$  less than a critical value  $\lambda_m(\beta)$  and no solution exists for  $\lambda > \lambda_m(\beta)$ . They also observe that triple solutions exist for  $0 < \beta \leq 0.14$ .

In the present article we use the relativistic similarity transformation presented in [1] to derive the similarity differential equation that governs the flow. This will be accomplished in the next section. The question of existence and non-uniqueness will be addressed in section III, and section IV will be devoted to the special case  $\beta = -1$ . Concluding remarks and a summary of the results will be presented in section V.

## 2 Formulation of the problem

The flow phenomena under consideration can be modeled by the Navier-Stokes equations:

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = U \frac{\partial U}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2} \quad (4)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (5)$$

subject to the boundary conditions

$$u(x, 0) = -\lambda U(x), \quad v(x, 0) = 0, \quad y \rightarrow \infty, \quad u \rightarrow U_\infty, \quad (6)$$

where  $u$  is the component of the velocity in the direction of the fluid flow, and  $v$  is the velocity in the direction normal to  $u$ . The constant  $\nu$  is the kinematic viscosity and  $U(x) = U_\infty x^m$ , where  $m$  is related to the constant  $\beta$  by the relation

$$\beta = \frac{2m}{m+1}.$$

To make the equations dimensionless, we follow the procedure presented in [9], and introduce the relativistic dimensionless coordinate  $\eta$  (see also [1] for details) such that for  $\lambda \neq -1$

$$\eta = y \sqrt{\frac{m+1}{2} \frac{|1+\lambda| U_\infty}{\nu} x^{m-1}}. \quad (7)$$

The equation of continuity (4) can be integrated by introducing a stream function  $\Psi(x, y)$  given by

$$\Psi(x, y) = \sqrt{\frac{2}{m+1}} \sqrt{\frac{\nu x U_\infty}{|1+\lambda|} x^{m+1}} f(\eta), \quad (8)$$

where  $f(\eta)$  denotes the dimensionless stream function. Thus the velocity components become

$$u = \frac{\partial \Psi}{\partial y} = U_\infty f'(\eta), \quad (9)$$

$$v = -\frac{\partial \Psi}{\partial x} = -\sqrt{\frac{m+1}{2}} \sqrt{\frac{U_\infty \nu x^{m-1}}{|1+\lambda|}} \left\{ f + \frac{m-1}{m+1} \eta f' \right\}. \quad (10)$$

Writing down the remaining terms of equation (3) and after some simplification the following ordinary differential equation will result

$$|1+\lambda| f''' + f f'' + \beta(1 - f'^2) = 0 \quad (11)$$

with the boundary conditions

$$f(0) = 0, \quad f'(0) = -\lambda, \quad f'(\infty) = 1 \quad (12)$$

Our goal now is to study the existence and nonuniqueness of solution of the modified Falkner-Skan Equation (11) with the initial and boundary conditions given by Equation (12).

### 3 Non-uniqueness of Solution

When  $\beta = 0$  the problem is reduced to the Blasius case. The existence and nonuniqueness of solution of this problem were discussed by Allan and Abu



Saris [3]. It was shown that two solutions exist on the range  $0 < \lambda < \lambda_c$ , where  $\lambda_c$  was found to be 0.3546... , one solution exists for  $\lambda = \lambda_c$  , and no solution exists for  $\lambda > \lambda_c$ . For the problem at hand we follow the same numerical techniques employed in [3] and solve the following initial value problem

$$|1 + \lambda|f''' + ff'' + \beta(1 - f'^2) = 0, \quad (13)$$

with the initial conditions

$$f(0) = 0, \quad f'(0) = -\lambda, \quad f''(0) = \alpha, \quad (14)$$

over the range of  $|\beta| < 1$ . Shooting method is employed to solve the above initial value problem and the value of  $f'(\eta)$  as  $\eta \rightarrow \infty$  is observed while the parameter  $\alpha$  is changed. The solution of the boundary value problem Equations (11) and (12) exists if there exists  $\alpha$  such that  $f'(\eta) \rightarrow 1$  as  $\eta \rightarrow \infty$ .

While carrying out the numerical calculations, it is observed that as  $\alpha \rightarrow 0$ ,  $f'(\infty)$  fluctuates very rapidly which may result in infinitely many solutions for the boundary value problem Equations (11) and (12). To overcome this problem we introduce the stretching transformation  $\eta = \frac{\zeta}{\epsilon}$  which transforms the initial value problem Equations (13) and (14) into the following alternative initial value problem

$$\epsilon|1 + \lambda|g'''(\zeta) + g(\zeta)g''(\zeta) + \beta(\frac{1}{\epsilon^2} - g'^2(\zeta)) = 0 \quad (15)$$

with the initial conditions

$$g(0) = 0, \quad g'(0) = -\frac{\lambda}{\epsilon}, \quad g''(0) = \frac{\alpha}{\epsilon^2}. \quad (16)$$

The results of the numerical calculation of solutions of Equations (15) and (16) for  $\lambda = 0.354..$  are displayed in Figure 1, in  $[\alpha, f'(\infty)]$  coordinates. One may conclude from that figure that given a value of  $\beta$  there exists a value  $\lambda_c(\beta)$  such that two solutions exist for all  $0 < \lambda < \lambda_c(\beta)$  , one solution exists for  $\lambda = \lambda_c(\beta)$  and no solution exists for  $\lambda > \lambda_c(\beta)$ .

## 4 The case $\beta = -1$

If  $\beta = -1$ , then Equation (11) is reduced to:

$$|1 + \lambda|f''' + ff'' + f'^2 - 1 = 0, \quad (17)$$

or

$$|1 + \lambda|f''' + (ff')' - 1 = 0. \quad (18)$$

Integrating Equation (18) from 0 to  $\eta$  leads to

$$|1 + \lambda|[f'' - \alpha] + ff' = \eta, \quad (19)$$

and integrating Equation (19) again from 0 to  $\eta$  and using the initial conditions at 0 lead to the following Ricatti type equation:

$$|1 + \lambda|[f' + \lambda - \alpha\eta] + \frac{f^2}{2} = \frac{\eta^2}{2} \quad (20)$$

Since  $f'(\eta) \rightarrow 1$  as  $\eta \rightarrow \infty$  one can assume that as  $\eta \rightarrow \infty$ ,  $f(\eta) \approx \eta + C$ . Then substituting the value of  $f$  in Equation (18) yields  $C = (1 + \lambda)\alpha$ . Then another evaluation of Equation (18) in the asymptotic limit as  $\eta \rightarrow \infty$  gives the relation

$$\alpha^2 = -2 \frac{(1 + \lambda)}{|1 + \lambda|},$$

which proves the following theorem.

*Theorem 3.1:* If  $\beta = -1$  then Equations (11) and (12) has a solution only if  $\lambda < -1$ .

In this case it is clear that if  $\lambda < -1$ , then  $f''(0) = \alpha = \pm\sqrt{2}$  and therefore two solutions exist.

However, if the condition at 0 is replaced by the condition  $f''(0) = \gamma \neq 0$ , then analysis similar to the above leads to the following relation between  $\alpha$  and  $\gamma$

$$\alpha = \frac{\gamma\lambda}{|1 + \lambda|} \pm \frac{\sqrt{\gamma^2 - 2(1 + \lambda)|1 + \lambda|}}{|1 + \lambda|}.$$

Thus, two solutions exist if  $\gamma^2 \geq 2(1 + \lambda)|1 + \lambda|$ . The solution in this case is presented in Figure (2). In that figure the two solutions subject to the

two choices of  $\alpha$  are displayed. It is clear from that Figure that when  $\alpha = \frac{-\sqrt{\gamma^2 - 2(1+\lambda)^2 + \gamma\lambda}}{|1+\lambda|}$ , a reverse flow ( $f'(\eta) < 0$ ) for some interval  $0 < \eta < \eta_1$  exists and also these solutions exhibit overshoot for almost all values of  $\lambda$ . However these two properties are not satisfied by the other set of solutions when  $\alpha = \frac{\sqrt{\gamma^2 - 2(1+\lambda)^2 + \gamma\lambda}}{|1+\lambda|}$ .

## 5 Conclusion:

In this article, the impact of the relativistic similarity transformation on the Falkner-Skan Equation has been discussed. It has been shown that if  $f(0) = 0$  then for a fixed value of  $\beta$ ,  $|\beta| < 1$ , two solutions exist for  $0 < \lambda < \lambda_c(\beta)$  and no solution exists for  $\lambda > \lambda_c(\beta)$ , while only one solution exists if  $\lambda = \lambda_c(\beta)$ . If  $\beta = -1$  it has been shown that two solutions exist, if  $\lambda < -1$ . If  $f(0) = \gamma \neq 0$ , it has been also shown that two solutions exist if  $\gamma^2 \geq 2(1+\lambda)|1+\lambda|$ .

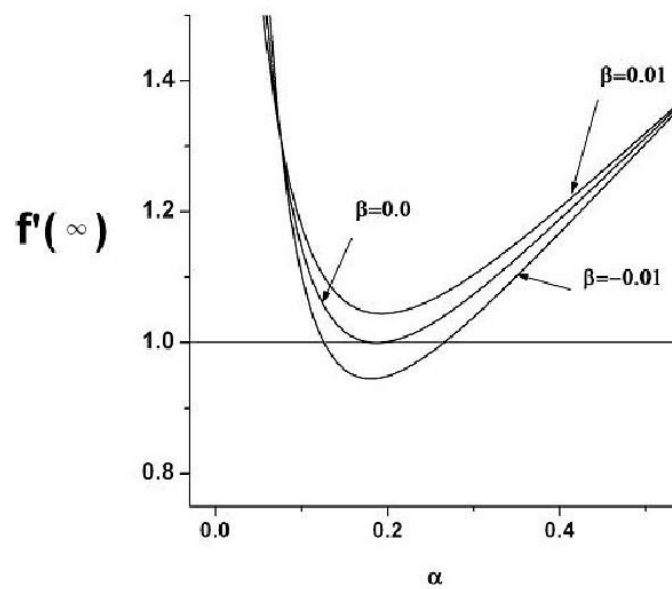


Figure 1:

The parameter space  $[\alpha = f''(0), f''(\infty)]$  for  $\lambda = 0.354$ , the for three values  $\beta = -0.01, \beta = 0.0$  and  $\beta = 0.01$ .

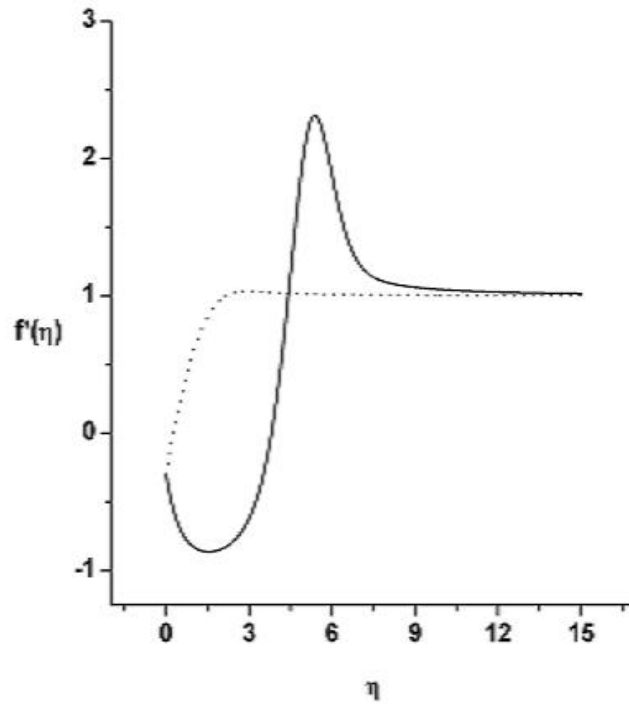


Figure 2:

The solution  $f'(\eta)$  for the case  $\beta = -1$  and  $f(0) = \gamma = 2(1 + \lambda)^2 + 0.03$  where  $\lambda = 0.3$ . The dotted line is the solution subject to the condition  $\alpha = \frac{\sqrt{\gamma^2 - 2(1+\lambda)^2 + \gamma\lambda}}{1+\lambda}$  and the continuous line is the solution subject to the condition  $\alpha = \frac{-\sqrt{\gamma^2 - 2(1+\lambda)^2 + \gamma\lambda}}{1+\lambda}$

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# On the construction of de la Vallée Poussin means for orthogonal polynomials using convolution structures

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## Abstract

In this paper we construct a de la Vallée Poussin approximation process for orthogonal polynomial expansions. Our construction is based on convolution structures which are established by the orthogonal polynomial system. We show that our approach leads to a natural generalization of the de la Vallée Poussin approximation process known from the trigonometric case. Finally we consider Jacobi polynomials and the generalized Chebyshev polynomials expansions as examples.

**Keywords:** Orthogonal polynomials, summation methods, de la Vallée Poussin kernel

**MSC:** 42C15, 33C45, 42C05

## 1 Introduction

Approximation processes for functions defined on the torus  $T = \{z \in \mathbb{C} : |z| = 1\}$  are well understood and an extensive literature exists on this subject. We mention here only the monographs of Butzer and Nessel [4] and Zygmund [26]. Many of this approximation processes are given by summation methods for the Fourier series. The most prominent summation processes are those of Fejér and de la Vallée Poussin. It is an essential fact that the Fejér and the de la Vallée Poussin approximation are given as a convolution of the function  $f$  to be approximated with a kernel  $K_n(z) = \sum_{k=-n}^n a_{n,k} z^k$ . Here the group structure of  $T$  is applied. The kernel  $K_n$  is usually defined as an average of the Dirichlet kernels  $D_n(z) = \sum_{k=-n}^n z^k$ . But in fact also the existence of a convolution structure on  $\mathbb{Z}$ , the dual of  $T$ , can be used in defining the coefficients of the corresponding kernel. Sometimes not very much attention is paid to this relation but it is indeed a crucial observation for us.

In this paper we are concerned with orthogonal Fourier expansions on a compact set  $S$  of the real line, where the basic functions are orthogonal polynomials  $\{P_n : n \in \mathbb{N}_0\}$ .

For our investigations we restricted ourselves to those orthogonal polynomials systems which establish a convolution structure on the set  $\mathbb{S}$  and on the natural numbers  $\mathbb{N}_0$ . These polynomial systems are related to the concept of hypergroups, see [3],[14] for more details.

Many results of mean convergence for orthogonal polynomial expansions exist. Most of them are restricted to Fourier-Jacobi expansions. We mention here the fundamental work of Pollard [18],[19],[20] and the books of Szegő [21] and Freud [9]. For the problem of best approximation we refer to Butzer, Jansche and Stens [5]. Saturation problems for Fourier-Jacobi expansions were studied by Bavinck [2], see also Yadav [25] and Li[17]. We would like to mention here also the recently published first volume of the collected works of Charles-Jean de la Vallée Poussin edited by Butzer, Mawhin and Vetro [7].

The aim of this paper is the construction of a de la Vallée Poussin approximation process for orthogonal polynomial expansions by using convolution structures. We show that in this case the process is again given as a convolution of the function  $f$  with a kernel  $K_n(x) = \sum_{k=0}^n a_{n,k} P_k(x)$ , where the coefficients  $a_{n,k}$  are given as a convolution of some special sequences. We prove that the operator  $\mathcal{K}_n f = K_n \star f$  shares quite a lot of the nice properties known from the trigonometric case. We mention here that it is also possible to construct a de la Vallée Poussin kernel by averaging Dirichlet kernels  $D_n(x) = \sum_{k=0}^n P_k(x)$ , see [24], [6]. But in contrast to the trigonometric case this approach leads not to the same kernel.

The contents of the paper are as follows. After recalling in section 2 some well known facts about the trigonometric case we develop in section 3 the main statements for the algebraic case. In section 4 we give some examples.

## 2 The trigonometric case

The aim of this section is to collect some basic facts on the de la Vallée Poussin operator for the trigonometric case. The main reason for presenting these classical statements is to make clear that the generalization for the algebraic case, which we are going to develop in the next section, is a very natural one. All the facts presented in this section are well known and can be found in many textbooks like [4],[15] and [26].

Let  $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ . By  $B$  we denote one of the Banach spaces  $(L^p(\mathbb{T}), \|\cdot\|_p)$ ,  $1 \leq p < \infty$  respectively  $(C(\mathbb{T}), \|\cdot\|_\infty)$ . The set of all trigonometric polynomials with degree at most  $n$  will be described by  $\mathcal{T}_n$ . The  $k$ -th Fourier coefficient of a function  $f \in B$  is given by

$$\check{f}(k) = \frac{1}{2\pi} \int_{\mathbb{T}} f(z) z^{-k} dz.$$

We define a sequence of operators by

$$\mathcal{A}_n f(z) = \sum_{k \in \mathbb{Z}} a_{n,k} \check{f}(k) z^k, \quad n \in \mathbb{N}_0, \quad (1)$$

where the coefficients  $a_{n,k}$  form a triangular scheme, i.e.,  $a_{n,k} = a_{n,-k}$ ,  $k = 0, \dots, n$ ,  $a_{n,k} = 0$ ,  $k > n$  and  $a_{n,0} = 1$ .

From these assumptions we have  $\mathcal{A}_n : B \rightarrow \mathcal{T}_n \subset B$  for every  $n \in \mathbb{N}_0$ .

Since there is a convolution structure on the torus  $\mathbb{T}$  the operator  $\mathcal{A}_n$  can be written as a convolution operator

$$\mathcal{A}_n f = A_n \star f, \quad (2)$$

where the kernel  $A_n$  is given by  $A_n(z) = \sum_{k=-n}^n a_{n,k} z^k$ .

From (2) we obtain  $\|\mathcal{A}_n f\|_B \leq \|A_n\|_1 \|f\|_B$ . Therefore the operator norm  $\|\mathcal{A}_n\|$  is bounded by the  $L^1$ -norm of the kernel.

As a consequence of the Banach-Steinhaus-Theorem we have

$$\lim_{n \rightarrow \infty} \|\mathcal{A}_n f - f\|_B = 0$$

if and only if

- (i)  $\lim_{n \rightarrow \infty} a_{n,k} = 1$  for all  $|k| \leq n$ ,
- (ii) there is a constant  $C > 0$  independent of  $n$  such that  $\|\mathcal{A}_n\| \leq C$  for all  $n \in \mathbb{N}_0$ .

One of the most important operators of this type is the de la Vallée Poussin operator  $\mathcal{V}_m^n$ , which is defined by a triangular scheme of the form

$$a_{m,k}^n = \begin{cases} 1 & |k| \leq n - m, \\ \frac{n+m+1-|k|}{2m+1} & \text{if } n - m + 1 \leq |k| < n + m + 1, \\ 0 & \text{otherwise,} \end{cases}$$

where  $m, n$  are non-negative integers with  $m \leq n$ .

We mention here two special cases of the de la Vallée Poussin operator:

- (i) For  $m = 0$  we get the partial sum operator

$$\mathcal{V}_0^n f = S_n f(z) = D_n \star f,$$

where  $D_n(z) = \sum_{k=-n}^n z^k$  is the Dirichlet kernel.

- (ii) For  $n = m$  we obtain the Cesàro operator

$$\mathcal{V}_n^n f = \sigma_{2n} f = F_{2n} \star f,$$

where  $F_n(z) = \sum_{k=-n}^n (1 - \frac{|k|}{n+1}) z^k = \frac{1}{n+1} \sum_{k=0}^n D_k(z)$  is the Fejér kernel.

By direct computation one can easily show that

$$V_m^n(z) = \frac{1}{2m+1} \sum_{k=n-m}^{n+m} D_k(z) = \frac{n+m+1}{2m+1} F_{n+m}(z) - \frac{n-m}{2m+1} F_{n-m-1}(z). \quad (3)$$

The latter part of equation (3) in combination with the well known fact  $\|F_n\|_1 = 1$  yields the estimate  $\|V_m^n\|_1 \leq \frac{2n+1}{2m+1}$ . Therefore the operator norm  $\|\mathcal{V}_m^n\|$  is uniformly bounded with respect to  $n, m$  provided that  $\sup_{n,m \in \mathbb{N}_0} \frac{2n+1}{2m+1} < \infty$ . This can easily be managed if we set  $n = m + p$  for a fixed  $p \in \mathbb{N}_0$ .

The crucial point here is that the kernel  $V_m^n$  can be written with the help of the convolution on  $\mathbb{Z}$ . The convolution of two sequences  $a, b \in \ell^1(\mathbb{Z})$  is defined as

$$a * b(n) = \sum_{k \in \mathbb{Z}} a(k)b(n-k), \quad n \in \mathbb{Z},$$

and the Fourier transform of  $a \in \ell^1(\mathbb{Z})$  is given by

$$\widehat{a}(z) = \sum_{k \in \mathbb{Z}} a(k) z^k, \quad z \in \mathbb{T}.$$

Let  $\chi_{\pm n}(k) = \begin{cases} 1 & |k| \leq n, \\ 0 & |k| > n \end{cases}$ . Using this sequence we can write

$$V_m^n(z) = \sum_{k \in \mathbb{Z}} \frac{\chi_{\pm n} * \chi_{\pm m}(k)}{\chi_{\pm n} * \chi_{\pm m}(0)} z^k = \frac{(\chi_{\pm n} * \chi_{\pm m})^\wedge(z)}{\chi_{\pm n} * \chi_{\pm m}(0)}.$$

It is indeed an easy exercise to show that  $\frac{\chi_{\pm n} * \chi_{\pm m}(k)}{\chi_{\pm n} * \chi_{\pm m}(0)} = a_{m,k}^n$ , where  $a_{m,k}^n$  are the coefficients defined above.

### 3 The algebraic case

The objective of this section is the construction of a de la Vallée Poussin operator for orthogonal polynomial expansions. The operator which we are going to construct should be a natural generalization of the de la Vallée Poussin operator for the trigonometric case.

One can try to use the same coefficients  $a_{m,k}^n$  as in the trigonometric case but it turns out that this is not the right choice. The reason for this is that these coefficients are intimately related to the orthogonal system  $(e^{ikt})_{k \in \mathbb{Z}}$ . If we work with orthogonal polynomial systems we have to invent something new. From the harmonic analysis point of view all this depends on the underlying convolution structure. Therefore we have to deal with convolution structures which are related to the orthogonal polynomial system as the usual convolution structures on  $\mathbb{Z}$  resp.  $\mathbb{T}$  are related to the system  $(e^{ikt})_{k \in \mathbb{Z}}$ . This leads to the so-called polynomial hypergroups, see [14], [3].

Let  $(a_n)_{n \in \mathbb{N}_0}, (c_n)_{n \in \mathbb{N}}$  be sequences of non-zero real numbers,  $(b_n)_{n \in \mathbb{N}_0}$  be a sequence of real numbers with the properties

$$\begin{aligned} a_0 + b_0 &= 1 \\ a_n + b_n + c_n &= 1. \end{aligned} \tag{4}$$

Let  $(R_n)_{n \in \mathbb{N}_0}$  be a polynomial sequence defined by

$$\begin{aligned} R_0(x) &= 1, \quad R_1(x) = \frac{1}{a_0}(x - b_0) \\ R_1(x)R_n(x) &= a_n R_{n+1}(x) + b_n R_n(x) + c_n R_{n-1}(x). \end{aligned} \tag{5}$$

The Theorem of Favard states that there is a probability measure  $\pi$  on  $\mathbb{R}$  such that the polynomials  $R_n$  are orthogonal with respect to  $\pi$ , i.e.,

$$\int_{\mathbb{R}} R_n(x) R_m(x) d\pi(x) = \delta_{n,m} \frac{1}{h(n)},$$

where  $h(n) = \|R_n\|_2^{-2}$ . The  $h(n)$ 's are usually called Haar weights. By  $\mathbb{S}$  we denote the support of the orthogonalization measure  $\pi$ .

We point out that in view of (4) we have an orthogonal polynomial system with

$$R_n(1) = 1. \quad (6)$$

Therefore the polynomials are not orthonormal. The corresponding orthonormal polynomial sequence  $(p_n)_{n \in \mathbb{N}_0}$  is given by  $p_n(x) = \sqrt{h(n)} R_n(x)$ .

The Haar weights can be represented in terms of the coefficients  $a_n$  and  $c_n$ . More precisely we have

$$h(0) = 1, \quad h(1) = \frac{1}{c_1}, \quad h(n) = \frac{a_1 \cdots a_{n-1}}{c_1 \cdots c_{n-1} c_n}, \quad n \geq 2. \quad (7)$$

We assume that the coefficients  $g(m, n, k)$  in the linearization formula

$$R_m(x) R_n(x) = \sum_{k=|n-m|}^{n+m} g(m, n, k) R_k(x) \quad (8)$$

are non-negative.

There are a lot of orthogonal polynomial systems which share this so-called non-negative linearization property. Among them are certain Jacobi polynomials, associated ultraspherical polynomials, generalized Chebyshev polynomials, Bernstein-Szegő polynomials to mention only a few. For more examples see [14].

Since the polynomials are normalized by (6) we have for the linearization coefficients the relation

$$\sum_{k=|n-m|}^{n+m} g(m, n, k) = 1. \quad (9)$$

One can show that the set  $\mathbb{D} = \{x \in \mathbb{R} : |R_n(x)| \leq 1 \text{ for all } n \in \mathbb{N}_0\}$  is a compact set which contains the set  $\mathbb{S}$ . In general the sets  $\mathbb{S}$  and  $\mathbb{D}$  need not to be equal, see [8]. Although in many important cases these sets coincide. So, we may assume  $\mathbb{S} = \mathbb{D}$ .

By  $B$  we denote one of the spaces  $L^p(\mathbb{S}, \pi)$ ,  $1 \leq p < \infty$  resp.  $C(\mathbb{S})$ . The set of all algebraic polynomials of degree at most  $n$  is denoted by  $\mathcal{P}_n$ . Since  $\mathbb{S}$  is compact the polynomials are dense in  $B$ .

The  $k$ -th Fourier coefficient of a function  $f \in B$  with respect to the orthogonal polynomial system  $(R_n)_{n \in \mathbb{N}_0}$  is given by

$$\check{f}(k) = \int_{\mathbb{S}} f(y) R_k(y) d\pi(y).$$

It is an important fact that those orthogonal polynomials which have the non-negative linearization property provide a convolution structure on the natural numbers in the

following way. As a generalized translation operator we define

$$T_m a(n) = \sum_{k=|n-m|}^{n+m} g(m, n, k) a(k), \quad (10)$$

where  $n, m \in \mathbb{N}_0$  and  $a$  is an element of the space  $\ell^1(\mathbb{N}_0, h)$ . The convolution of two sequences  $a, b \in \ell^1(\mathbb{N}_0, h)$  is defined as

$$a \star b(n) = \sum_{k=0}^{\infty} a(k) T_k b(n) h(k). \quad (11)$$

With this convolution the Banach space  $\ell^1(\mathbb{N}_0, h)$  becomes a commutative Banach algebra. The Fourier transform of a sequence  $a \in \ell^1(\mathbb{N}_0, h)$  is given by

$$\widehat{a}(x) = \sum_{k=0}^{\infty} a(k) R_k(x) h(k), \quad x \in \mathbb{S}.$$

In order to construct the de la Vallée Poussin operator we consider the sequence

$$\chi_n(k) = \begin{cases} 1 & k \leq n, \\ 0 & k > n. \end{cases}$$

Let  $n, m \in \mathbb{N}_0$  with  $n \geq m$ . We now consider the convolution product  $\chi_n \star \chi_m$ . In the next Proposition we collect some properties of this sequence.

**Proposition 3.1** *Let  $n, m \in \mathbb{N}_0$  with  $n \geq m$ . For the sequence  $\chi_n \star \chi_m$  we have*

$$(i) \quad \chi_n \star \chi_m(k) = \sum_{j=0}^m h(j) \quad \text{for } 0 \leq k \leq n - m,$$

$$(ii) \quad \chi_n \star \chi_m(k) = \sum_{j=0}^m \sum_{l=|k-j|}^{\min(n, k+j)} g(j, k, l) h(j) \quad \text{for } n - m < k \leq n + m,$$

$$(iii) \quad \chi_n \star \chi_m(n + m) = g(n + m, m, n) h(m),$$

$$(iv) \quad \chi_n \star \chi_m(k) = 0 \quad \text{for } k > n + m.$$

PROOF: By the equations (10) and (11) we obtain

$$\chi_n \star \chi_m(k) = \sum_{j=0}^{\infty} \chi_n(j) T_j \chi_m(k) h(j) = \sum_{j=0}^m \sum_{l=|k-j|}^{k+j} g(j, k, l) \chi_n(l) h(j).$$

If  $0 \leq k \leq n - m$  we have  $0 \leq k + j \leq n$  and therefore it follows in view of (9)

$$\sum_{l=|k-j|}^{k+j} g(j, k, l) \chi_n(l) = \sum_{l=|k-j|}^{k+j} g(j, k, l) = 1.$$

This shows (i). (ii) and (iii) follow directly from the definition. If  $k > n + m$  we get  $k - j \leq n$  and therefore

$$\sum_{l=k-j}^{k+j} g(j, k, l) \chi_n(l) = 0 .$$

This proves (iv). ◇

We now define the de la Vallée Poussin operator with the help of the sequence  $\chi_n \star \chi_m$ . Let  $f \in B$  and let  $n, m \in \mathbb{N}_0$  be natural numbers with  $n \geq m$ . Let

$$a_{m,k}^n = \frac{\chi_n \star \chi_m(k)}{\chi_n \star \chi_m(0)} . \quad (12)$$

Then the operator is defined as

$$\mathcal{V}_m^n f(x) = \sum_{k=0}^{\infty} a_{m,k}^n \check{f}(k) R_k(x) h(k) . \quad (13)$$

Obviously, the operator is linear and maps  $B$  into  $\mathcal{P}_{n+m}$ . Using the definition of the Fourier coefficient the operator can be written as

$$\mathcal{V}_m^n f(x) = \int_{\mathbb{S}} f(y) K_m^n(x, y) d\pi(y), \quad (14)$$

where the kernel is given by  $K_m^n(x, y) = \sum_{k=0}^{n+m} a_{m,k}^n R_k(x) R_k(y) h(k)$ .

Moreover, we have the following invariance property.

**Proposition 3.2** *For the de la Vallée Poussin operator we have*

$$\mathcal{V}_m^n p = p \quad \text{for every } p \in \mathcal{P}_{n-m} .$$

PROOF: Let  $p(x) = \sum_{j=0}^{n-m} c_j R_j(x)$ . Since

$$\check{p}(k) = \int_{\mathbb{S}} p(x) R_k(x) d\pi(x) = \frac{1}{h(k)} c_k$$

we get  $\mathcal{V}_m^n p = p$  by Proposition 3.1(i). ◇

Two special cases of the de la Vallée Poussin operator are worthwhile to mention.

**Remark 3.3** (i) *If we set  $m = 0$  we get back the partial sum operator*

$$\mathcal{S}_n f(x) = \sum_{k=0}^n \check{f}(k) R_k(x) h(k)$$

*as we can see immediately from Proposition 3.1.*

(ii) In the case  $m = n$  we have a Cesàro operator of the form

$$\sigma_{2n}f(x) = \sum_{k=0}^{2n} a_{n,k}^n \check{f}(k) R_k(x) h(k),$$

which is a natural generalization of the Cesàro operator for the trigonometric case. One should note that we get in this way only Cesàro operators of even order.

Now we are going to derive conditions for the operator to have the approximation property

$$\lim_{n,m \rightarrow \infty} \|\mathcal{V}_m^n f - f\|_B = 0. \quad (15)$$

The fact that the polynomials are dense in  $B$  in combination with the invariance property gives us (15) provided that the operator norm  $\|\mathcal{V}_m^n\|_{B \rightarrow B}$  is uniformly bounded with respect to  $n$  and  $m$ . To prove this one can try to estimate the expression  $\sup_{n,m \in \mathbb{N}_0} \|K_m^n\|_\infty$  which is in general not an easy task. The estimation of the operator norm is easier to achieve if there is a convolution structure on the set  $\mathbb{S}$ . This convolution is established by a so-called product formula of the type

$$R_n(x)R_n(y) = \int_{\mathbb{S}} R_n(z) d\mu_{x,y}(z), \quad (16)$$

where  $\mu_{x,y}$  is a probability measure on  $\mathbb{S}$  for all  $x, y \in \mathbb{S}$ . This product formula gives us a translation operator

$$T_y f(x) = \int_{\mathbb{S}} f(z) d\mu_{x,y}(z), \quad (17)$$

where  $f \in C(\mathbb{S})$ . It can be shown that this operator can be extended to the spaces  $L^p(\mathbb{S}, \pi)$ ,  $1 \leq p \leq \infty$ . Moreover we have  $\|T_x f\|_B \leq \|f\|_B$  for all  $x \in \mathbb{S}$  and  $f \in B$ , see [3]. It is an important fact that the orthogonalization measure  $\pi$  is invariant with respect to this generalized translation, i.e.,

$$\int_{\mathbb{S}} T_x f(y) d\pi(y) = \int_{\mathbb{S}} f(y) d\pi(y)$$

for all  $x \in \mathbb{S}$  and  $f \in B$ . Furthermore we have

$$\int_{\mathbb{S}} f(y) T_x g(y) d\pi(y) = \int_{\mathbb{S}} T_x f(y) g(y) d\pi(y). \quad (18)$$

The convolution of functions  $f, g \in L^1(\mathbb{S}, \pi)$  is defined as

$$f * g(x) = \int_{\mathbb{S}} f(y) T_y g(x) d\pi(y). \quad (19)$$

Equation (18) shows the commutativity of this convolution product. Using standard arguments one can prove that for  $f \in L^1(\mathbb{S}, \pi)$ ,  $g \in B$  we obtain  $f * g \in B$  and  $\|f * g\|_B \leq \|f\|_1 \|g\|_B$ .

If  $(R_n)_{n \in \mathbb{N}_0}$  is an orthogonal polynomial system for which a non-negative linearization and a product formula holds we are able to estimate the norm  $\|\mathcal{V}_m^n\|_{B \rightarrow B}$ .



**Theorem 3.4** Assume that for the orthogonal polynomial system  $(R_n)_{n \in \mathbb{N}_0}$  the non-negative linearization and a product formula holds. Then the norm of the de la Vallée Poussin operator  $\|\mathcal{V}_m^n\|_{B \rightarrow B}$  is uniformly bounded with respect to  $m$  and  $n$  if

$$\sup_{n, m \in \mathbb{N}_0} \frac{\sum_{j=0}^n h(j)}{\sum_{j=0}^m h(j)} < \infty. \quad (20)$$

PROOF: Since for  $(R_n)_{n \in \mathbb{N}_0}$  we have non-negative linearization and a product formula the de la Vallée Poussin operator can be represented as a convolution operator, i.e.,

$$\mathcal{V}_m^n f(x) = V_m^n * f(x) = \int_{\mathbb{S}} V_m^n(y) T_y f(x) d\pi(y), \quad (21)$$

where  $V_m^n(x) = \sum_{k=0}^{n+m} a_{m,k}^n R_k(x) h(k)$  and the  $a_{m,k}^n$ 's are defined as above. Therefore we obtain estimation  $\|\mathcal{V}_m^n\|_{B \rightarrow B} \leq \|V_m^n\|_1$ .

By Proposition 3.1 we have

$$\|V_m^n\|_1 = \frac{1}{\sum_{j=0}^m h(j)} \int_{\mathbb{S}} |(\chi_n \star \chi_m)^\wedge(x)| d\pi(x) = \frac{1}{\sum_{j=0}^m h(j)} \int_{\mathbb{S}} |D_n(x) D_m(x)| d\pi(x),$$

where  $D_n(x) = \widehat{\chi_n}(x) = \sum_{k=0}^n R_k(x) h(k)$ .

Using the identity  $D_n(x) D_m(x) = \frac{1}{2} [D_n^2(x) + D_m^2(x) - (D_n(x) - D_m(x))^2]$  and the orthogonality relation we obtain the following estimate

$$\begin{aligned} 2 \int_{\mathbb{S}} |D_n(x) D_m(x)| d\pi(x) &\leq \int_{\mathbb{S}} \sum_{k,j=0}^n R_j(x) R_k(x) h(j) h(k) d\pi(x) \\ &+ \int_{\mathbb{S}} \sum_{k,j=0}^m R_j(x) R_k(x) h(j) h(k) d\pi(x) + \int_{\mathbb{S}} \sum_{k,j=m+1}^n R_j(x) R_k(x) h(j) h(k) d\pi(x) \\ &\leq \sum_{j=0}^n h(j) + \sum_{j=0}^m h(j) + \sum_{j=m+1}^n h(j) = 2 \sum_{j=0}^n h(j). \end{aligned}$$

This leads to

$$\|V_m^n\|_1 \leq \frac{\sum_{j=0}^n h(j)}{\sum_{j=0}^m h(j)}$$

This completes the proof.  $\diamond$

Our condition of the boundedness of the norm  $\|V_m^n\|_1$  is somehow related to the so-called property (H) of an orthogonal polynomial system. An orthogonal polynomial system has the property (H) if

$$\lim_{n \rightarrow \infty} \frac{h(n)}{\sum_{k=0}^n h(k)} = 0. \quad (22)$$

For our purpose we need the following well-known Lemma, see [16]. For sake of completeness we present the proof.

**Lemma 3.5** (i) *The condition (H) holds if*

$$\lim_{n \rightarrow \infty} \frac{c_n}{a_{n-1}} = 1.$$

(ii) *Let (H) be satisfied. Then for every fixed number  $\ell \in \mathbb{Z}$  we have*

$$\lim_{n \rightarrow \infty} \frac{h(n + \ell)}{\sum_{k=0}^n h(k)} = 0.$$

PROOF: (i) Let  $\gamma_n = \frac{c_n}{a_{n-1}}$ . For a given  $\varepsilon \in (0, 1)$  there is a  $m \in \mathbb{N}$  such that  $|\gamma_{n+m} - 1| < \varepsilon$  for all  $n \in \mathbb{N}$ . Using (7) we obtain

$$\begin{aligned} \frac{h(n+m)}{\sum_{j=0}^{n+m} h(j)} &\leq \frac{h(n+m)}{\sum_{j=m}^{n+m} h(j)} \\ &\leq \frac{h(n+m)}{h(n+m)[1 + \gamma_{n+m} + \gamma_{n+m}\gamma_{n+m-1} + \cdots + (\gamma_{n+m}\gamma_{n+m-1} \cdots \gamma_{m+1})]} \\ &\leq \frac{1}{1 + (1-\varepsilon) + (1-\varepsilon)^2 + \cdots + (1-\varepsilon)^n} = \frac{\varepsilon}{1 - (1-\varepsilon)^{n+1}}. \end{aligned}$$

This shows (i).

For  $\ell \leq 0$  the assertion follows directly from condition (H). Let  $\ell > 0$  be a fixed integer. Let  $0 < \varepsilon < \frac{1}{2\ell}$ . Then for sufficiently large  $n$  we have

$$\begin{aligned} \frac{h(n+\ell)}{\sum_{j=0}^n h(j)} &= \frac{h(n+\ell)}{\sum_{j=0}^{n+\ell} h(j)} \frac{\sum_{j=0}^{n+\ell} h(j)}{\sum_{j=0}^n h(j)} = \frac{h(n+\ell)}{\sum_{j=0}^{n+\ell} h(j)} \frac{\sum_{j=0}^{n+\ell} h(j)}{\sum_{j=0}^{n+\ell} h(j) - h(n+1) - \cdots - h(n+\ell)} \\ &< \frac{\varepsilon}{1 - \ell\varepsilon} < 2\varepsilon. \end{aligned}$$

This proves (ii). ◇

If we choose  $n = m + \ell$  in the construction of the de la Vallée Poussin operator, where  $\ell \geq 0$  is a fixed integer, we obtain

$$\|\mathcal{V}_m^n\|_{B \rightarrow B} \leq 1 + \frac{h(m+1) + \cdots + h(m+\ell)}{\sum_{k=0}^m h(k)} \quad (23)$$

Now Lemma 3.5 shows that  $\|\mathcal{V}_m^n\|_{B \rightarrow B}$  is uniformly bounded with respect to  $m$  provided that (H) holds.

In view of Proposition 3.2 and Theorem 3.4 we have immediately the following fact.

**Corollary 3.6** *Under the assumptions of Theorem 3.4 we have*

$$\|\mathcal{V}_m^n f - f\|_B \leq \left(1 + \frac{\sum_{j=0}^n h(j)}{\sum_{j=0}^m h(j)}\right) E_{n-m}(f)_B,$$

where  $E_{n-m}(f)_B$  is the error of the best approximation by polynomials from  $\mathcal{P}_{n-m}$ .

The polynomial  $D_n(x) = \sum_{k=0}^n R_k(x) h(k)$  is the Dirichlet kernel. Using the Christoffel-Darboux identity we get the explicit expression

$$D_n(x) = a_0 a_n h(n) \frac{R_{n+1}(x) - R_n(x)}{x - 1}. \quad (24)$$

From (24) we obtain

$$\sum_{k=0}^n h(k) = a_0 a_n h(n) (R'_{n+1}(1) - R'_n(1)). \quad (25)$$

The Dirichlet kernel is the so-called kernel polynomial for the orthogonal polynomial system  $(R_n)_{n \in \mathbb{N}_0}$ . Therefore the system  $(D_n)_{n \in \mathbb{N}_0}$  is an orthogonal polynomial system with respect to the measure  $d\pi^*(x) = (1-x)d\pi(x)$ . An easy calculation shows that this system fulfils a recurrence relation of the form

$$D_0(x) = 1, \quad D_1(x) = \frac{1}{a_0 c_1} (x - (b_0 - a_0 c_1)),$$

$$x D_n = a_0 c_{n+1} D_{n+1} + (1 - a_0(a_n + c_{n+1})) D_n(x) + a_0 a_n D_{n-1}(x), \quad n \geq 1.$$

Of course these polynomials are not normalized to be one at the point  $x = 1$ . One can ask whether the  $D_n$ 's have a non-negative linearization. We can derive a sufficient condition by using a criterion of Askey [1], see also [22],[23] for more details on non-negative linearization.

**Proposition 3.7** *The orthogonal polynomial system  $(D_n)_{n \in \mathbb{N}_0}$  has a non-negative linearization if*

$$a_n c_n \leq a_{n+1} c_{n+1} \quad \text{and} \quad c_{n+2} + a_{n+1} \leq c_{n+1} + a_n$$

for all  $n \geq 1$ .

PROOF: Since the corresponding monic polynomials  $\tilde{D}_n$  fulfil a recurrence relation of the form

$$x \tilde{D}_n = \tilde{D}_{n+1} + (1 - a_0(a_n + c_{n+1})) \tilde{D}_n + a_0^2 a_n c_n \tilde{D}_{n-1}$$

the assertion follows from the criterion of Askey [1].  $\diamond$

Since the  $D_n$ 's are orthogonal we obtain  $D_n D_m = \sum_{k=|n-m|}^{n+m} d(n, m; k) D_k$ . Hence we have for the de la Vallée Poussin kernel

$$V_m^n(x) = \frac{1}{\sum_{j=0}^m h(j)} \sum_{k=n-m}^{n+m} d(n, m; k) D_k(x). \quad (26)$$

In case that the  $D_n$ 's have a non-negative linearization the  $V_m^n$  is a weighted average of Dirichlet kernels and (26) generalizes the first part of (3). It is interesting to ask whether we can generalize also the second part of (3), i.e., can we write the kernel  $V_m^n$  as a linear combination of Fejér kernels. For such a formula we need a generalized Fejér kernel of odd order which we do not get in a natural way from our construction. Up to now it is not clear to us how to generalize the latter part of (3).

## 4 Examples

In this section we consider some concrete examples in detail. We concentrate on three examples namely the Jacobi polynomials, the generalized Chebyshev polynomials and the little  $q$ -Legendre polynomials.

### 4.1 Jacobi polynomials

The Jacobi polynomials  $R_n^{(\alpha, \beta)}$  fulfil a recurrence relation of the form (5) with coefficients

$$a_n = \frac{(n + \alpha + \beta + 1)(n + \alpha + 1)(\alpha + \beta + 2)}{(2n + \alpha + \beta + 2)(2n + \alpha + \beta + 1)(\alpha + 1)}, \quad c_n = \frac{n(n + \beta)(\alpha + \beta + 2)}{(2n + \alpha + \beta + 1)(2n + \alpha + \beta)(\alpha + 1)},$$

$$b_n = \frac{\alpha - \beta}{2(\alpha + 1)} \left[ 1 - \frac{(\alpha + \beta + 2)(\alpha + \beta)}{(2n + \alpha + \beta + 2)(2n + \alpha + \beta)} \right]$$

for  $n \geq 1$  and

$$a_0 = \frac{2(\alpha + 1)}{\alpha + \beta + 2}, \quad b_0 = \frac{\beta - \alpha}{\alpha + \beta + 2}.$$

Using (7) an easy computation gives us

$$h(0) = 1, \quad h(n) = \frac{(2n + \alpha + \beta + 1)(\alpha + \beta + 1)_n (\alpha + 1)_n}{(\alpha + \beta + 1)(\beta + 1)_n n!}.$$

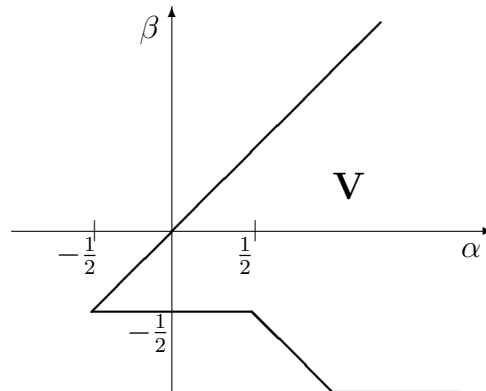


Fig.a: The set  $V$ .

In [10],[11] Gasper showed that for the Jacobi polynomials a non-negative linearization and a product formula holds, provided that the parameters  $(\alpha, \beta)$  belong to the set  $V = \{(\alpha, \beta) \in \mathbb{R}^2 : \alpha \geq \beta > -1; \beta \geq -\frac{1}{2} \text{ or } \alpha + \beta \geq 0\}$ . Using Lemma 3.5 it is easy to see that the Jacobi polynomials share the property (H). Therefore the norm of the de la Vallée Poussin operator  $\mathcal{V}_m^n$  generated by the Jacobi polynomials is bounded if we choose  $n = m + \ell$  for a fixed positive integer  $\ell$ .

Since

$$(R_n^{(\alpha, \beta)})'(1) = \frac{n(n + \alpha + \beta + 1)}{2(\alpha + 1)}$$

we are able to compute a bound explicitly by applying (24)

$$\frac{\sum_{j=0}^n h(j)}{\sum_{j=0}^m h(j)} = \frac{(n + \alpha + 1)(n + \alpha + \beta + 1)(\alpha + \beta + 1)_n (\alpha + 1)_n (\beta + 1)_m m!}{(m + \alpha + 1)(m + \alpha + \beta + 1)(\alpha + \beta + 1)_m (\alpha + 1)_m (\beta + 1)_n n!}. \quad (27)$$

We consider some special cases.

- a) For the Chebyshev polynomials of the first kind ( $\alpha = \beta = -\frac{1}{2}$ ) we get as a bound for the operator norm

$$\|\mathcal{V}_m^n\|_{B \rightarrow B} \leq \frac{2n + 1}{2m + 1},$$

which is the well-known result from the trigonometric case.

- b) For the Legendre polynomials ( $\alpha = \beta = 0$ ) we have the bound

$$\|\mathcal{V}_m^n\|_{B \rightarrow B} \leq \left( \frac{n + 1}{m + 1} \right)^2.$$

- c) For the Chebyshev polynomials of the second kind ( $\alpha = \beta = \frac{1}{2}$ ) we obtain

$$\|\mathcal{V}_m^n\|_{B \rightarrow B} \leq \frac{(n + 1)(n + 2)(2n + 3)}{(m + 1)(m + 2)(2m + 3)}.$$

Since the Dirichlet kernels  $D_n^{(\alpha, \beta)} = \sum_{k=0}^n R_k^{(\alpha, \beta)} h(k)$  are orthogonal with respect to the measure  $d\pi^*(x) = (1 - x)^{\alpha+1}(1 + x)^\beta \chi_{[-1, 1]} dx$  we get  $D_n^{(\alpha, \beta)} = P_n^{(\alpha+1, \beta)}$ , where  $P_n^{(\alpha+1, \beta)}$  is a Jacobi polynomial which is not normalized at the point  $x = 1$ . Therefore we have non-negative linearization for the polynomials  $D_n^{(\alpha, \beta)}$  whenever  $(\alpha, \beta) \in V$ . So the corresponding de la Vallée Poussin kernel is really a weighted average of Dirichlet kernels in this case.

## 4.2 Generalized Chebyshev polynomials

The generalized Chebyshev polynomials  $T_n^{(\alpha, \beta)}$  fulfil the recurrence relation (5) with coefficients

$$a_n = \begin{cases} \frac{\ell + \alpha + \beta + 1}{2\ell + \alpha + \beta + 1} & \text{if } n = 2\ell, \\ \frac{\ell + \alpha + 1}{2\ell + \alpha + \beta + 2} & \text{if } n = 2\ell + 1, \end{cases} \quad c_n = \begin{cases} \frac{\ell}{2\ell + \alpha + \beta + 1} & \text{if } n = 2\ell, \\ \frac{\ell + \beta + 1}{2\ell + \alpha + \beta + 2} & \text{if } n = 2\ell + 1 \end{cases}$$

and  $b_n = 0$  for  $n \geq 1$ ,  $a_0 = 1$  and  $b_0 = 0$ . The polynomials are orthogonal with respect to the measure  $d\pi(x) = (1-x^2)^\alpha |x|^{2\beta+1} \chi_{[-1,1]} dx$ .

For the Haar weights we obtain

$$h(0) = 1, \quad h(n) = \begin{cases} \frac{(2\ell+\alpha+\beta+1)(\alpha+\beta+1)_\ell(\alpha+1)_\ell}{(\alpha+\beta+1)(\beta+1)_\ell \ell!} & \text{if } n = 2\ell, \\ \frac{(2\ell+\alpha+\beta+2)(\alpha+\beta+2)_\ell(\alpha+1)_\ell}{(\beta+1)_{\ell+1} \ell!} & \text{if } n = 2\ell + 1. \end{cases}$$

For  $\beta \geq -\frac{1}{2}$ ,  $\alpha \geq \beta + 1$  the generalized Chebyshev polynomials have non-negative linearization and a product formula, see [13],[14].

Using again (25) we compute

$$\sum_{j=0}^n h(j) = \begin{cases} \frac{(\ell+\alpha+\beta+1)(2\ell+\alpha+1)(\alpha+\beta+1)_\ell(\alpha+1)_\ell}{(\alpha+\beta+1)(\alpha+1)(\beta+1)_\ell \ell!} & \text{if } n = 2\ell, \\ \frac{(\ell+\alpha+1)(2\ell+\alpha+2\beta+3)(\alpha+\beta+2)_\ell(\alpha+1)_\ell}{(\alpha+1)(\beta+1)_{\ell+1} \ell!} & \text{if } n = 2\ell + 1. \end{cases}$$

This expression shows that we can bound the quotient  $\sum_{j=0}^n h(j) / \sum_{j=0}^m h(j)$  uniformly with respect to  $m$  if we choose  $n = m + p$  for a fixed  $p \geq 0$ .

### 4.3 Little $q$ -Legendre polynomials

Finally we present an example for which the condition of Theorem 3.4 fails to hold. This example is given by the little  $q$ -Legendre polynomials. For a fixed  $q \in (0, 1)$  this polynomial system is defined by a recurrence relation with coefficients

$$a_0 = \frac{1}{1+q}, \quad b_0 = \frac{q}{1+q},$$

$$a_n = q^n \frac{(1+q)(1-q^{n+1})}{(1-q^{2n+1})(1+q^{n+1})}, \quad b_n = \frac{(1-q^n)(1-q^{n-1})}{(1+q^n)(1+q^{n+1})}, \quad c_n = q^n \frac{(1+q)(1-q^n)}{(1-q^{2n+1})(1+q^n)}.$$

The support of the orthogonality measure is the set  $\{1\} \cup \{1 - q^{2k} : k \in \mathbb{N}_0\}$ . For these orthogonal polynomials a non-negative linearization and a product formula hold, see [12]. Using (7) an easy calculation gives

$$h(n) = \frac{1}{1-q} \left( \frac{1}{q^n} - q^{n+1} \right).$$

So the  $h(n)$ 's grow exponentially. Therefore the quotient  $\frac{\sum_{k=0}^n h(k)}{\sum_{k=0}^m h(k)}$  can not be bounded uniformly with respect to  $n$  and  $m$ .

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# CONVERGENCE THEOREMS OF THE ISHIKAWA ITERATIVE SCHEME FOR ASYMPTOTICALLY PSEUDO-CONTRACTIVE MAPPINGS IN BANACH SPACES

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**ABSTRACT.** In this paper, some convergence theorems of the modified Ishikawa and Mann iterative schemes for asymptotically pseudo-contractive and asymptotically nonexpansive mappings in Banach space are obtained. The results presented in this paper improve and extend the corresponding results in Goebel and Kirk [5], Kirk [6], Liu [7] and Schu [9].

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## 1. Introduction and Preliminaries

Throughout this paper, we assume that  $E$  is a real Banach space,  $E^*$  is the topological dual space of  $E$  and  $\langle \cdot, \cdot \rangle$  is the dual pair between  $E$  and  $E^*$ . Let  $D(T)$

and  $F(T)$  denote the domain of  $T$  and the set of all fixed points of  $T$  respectively, and  $J : E \rightarrow 2^{E^*}$  be the normalized duality mapping defined by

$$J(x) = \{f \in E^* : \langle x, f \rangle = \|x\| \cdot \|f\|, \|f\| = \|x\|\}$$

for all  $x \in E$ .

**Definition 1.1.** Let  $T : D(T) \subset E \rightarrow E$  be a mapping.

(1) The mapping  $T$  is said to be *asymptotically nonexpansive* if there exists a sequence  $\{k_n\}$  in  $[1, \infty)$  with  $\lim_{n \rightarrow \infty} k_n = 1$  such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\|$$

for all  $x, y \in D$  and  $n = 1, 2, \dots$ .

(2) The mapping  $T$  is said to be *asymptotically pseudo-contractive* if there exists a sequence  $\{k_n\}$  in  $[1, \infty)$  with  $\lim_{n \rightarrow \infty} k_n = 1$  and, for any  $x, y \in D$ , there exists  $j(x - y) \in J(x - y)$  such that

$$\langle T^n x - T^n y, j(x - y) \rangle \leq k_n \|x - y\|^2$$

for all  $n = 1, 2, \dots$ .

The following proposition follows from Definition 1.1 immediately.

**Proposition 1.** (1) If  $T : D(T) \subset E \rightarrow E$  is a nonexpansive mappings, then  $T$  is an asymptotically nonexpansive mapping with a constant sequence  $\{1\}$ .

(2) If  $T : D(T) \subset E \rightarrow E$  is an asymptotically nonexpansive mapping, then  $T$  is asymptotically pseudo-contractive. But the converse is not true in general.

This can be seen from the following example.

**Example 1.1.** ([8]) Let  $E = \mathbb{R}$ ,  $D = [0, 1]$  and the mapping  $T : D \rightarrow D$  is defined by

$$Tx = (1 - x^{\frac{2}{3}})^{\frac{3}{2}}$$

for all  $x \in D$ . It can be proved that  $T$  is not Lipschitzian and so it is not asymptotically nonexpansive. Since  $T$  is monotonically decreasing and  $T \circ T = I$ , we have

$$\begin{aligned} & (T^n x - T^n y)(x - y) \\ &= \begin{cases} |x - y|^2 & \text{if } n \text{ is even,} \\ (Tx - Ty)(x - y) \leq 0 \leq |x - y|^2 & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

This implies that  $T$  is an asymptotically pseudo-contractive mapping with a constant sequence  $\{1\}$ .

**Definition 1.2.** (1) Let  $T : D(T) \subset E \rightarrow E$  be a mapping and  $\{\alpha_n\}, \{\beta_n\}$  be sequences in  $[0, 1]$ . Then the sequence  $\{x_n\}$  defined by

$$(1.1) \quad \begin{cases} x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n y_n, \\ y_n = (1 - \beta_n)x_n + \beta_n T^n x_n \end{cases}$$

for all  $n \geq 0$  is called the *modified Ishikawa iterative scheme* of  $T$ .

(2) In (1.1), if  $\beta_n = 0$  for  $n = 0, 1, 2, \dots$ , then  $y_n = x_n$ . The sequence  $\{x_n\}$  defined by

$$(1.2) \quad x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n$$

for all  $n \geq 0$  is called the *modified Mann iterative scheme* of  $T$ .

The concept of asymptotically nonexpansive mapping was introduced by Goebel and Kirk [5] in 1972, which was closely related to the theory of fixed points of mappings in Banach spaces. An early fundamental result due to Goebel and Kirk [5] showed that, if  $E$  is a uniformly convex Banach space,  $D$  is a nonempty bounded closed convex subset of  $E$  and  $T : D \rightarrow D$  is an asymptotically nonexpansive mapping, then  $T$  has a fixed point in  $D$ . This result is a generalization of the corresponding results in Browder [1] and Kirk [6].

On the other hand, the concept of asymptotically pseudo-contractive mapping was introduced by Schu [9] in 1991.

The iterative approximation problems for nonexpansive mapping, asymptotically nonexpansive mapping and asymptotically pseudo-contractive mapping were studied extensively by Browder [1], Goebel and Kirk [5], Kirk [6], Liu [7], Rhoades [8], Schu [9], Xu [10,11] and Xu and Roach [12] in the setting of Hilbert space or uniformly convex Banach spaces.

In this paper is, by using a new iterative technique, we study the iterative approximation problems of fixed points for asymptotically pseudo-contractive mappings and asymptotically nonexpansive mappings in uniformly smooth Banach space. The main results in this paper extend and improve the corresponding results in [5]–[7] and [9] and, further, the methods of proof given in this paper are also quite different from others.

The following lemmas play an important role in this paper:

**Lemma 1.1.** *Let  $E$  be a real Banach space and  $J : E \rightarrow 2^{E^*}$  be a normalized duality mapping. Then, for all  $x, y \in E$  and  $j(x + y) \in J(x + y)$ ,*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y) \rangle.$$

**Lemma 1.2.** *Let  $E$  be a real uniformly smooth Banach space. Then the normalized duality mapping  $J : E \rightarrow E^*$  is single-valued and uniformly continuous on any bounded subset of  $E$ .*

**Lemma 1.3.** [12] *Let  $E$  be a uniformly convex real Banach space,  $D$  be a nonempty closed convex subset of  $E$  and  $T : D \rightarrow D$  be a nonexpansive mapping. If  $F(T) \neq \emptyset$ , then, for any  $x \in D, q \in F(T)$  and  $j(x - q) \in J(x - q)$ , the equality*

$$\langle Tx - q, j(x - q) \rangle - \|x - q\|^2 = 0$$

*holds if and only if  $x = q$ .*

## II. The Main Results

Now, we give our main theorems in this paper.

**Theorem 2.1.** *Let  $E$  be a real uniformly smooth Banach space,  $D$  be a nonempty bounded closed convex subset of  $E$  and  $T : D \rightarrow D$  be an asymptotically pseudo-contractive mapping with a sequence  $\{k_n\} \subset [1, \infty)$ ,  $\lim_{n \rightarrow \infty} k_n = 1$  and  $F(T) \neq \emptyset$ . Let  $\{\alpha_n\}$  and  $\{\beta_n\}$  be two real sequences in  $[0, 1]$  satisfying the following conditions:*

- (i)  $\alpha_n \rightarrow 0, \beta_n \rightarrow 0 \quad (n \rightarrow \infty)$ ,
- (ii)  $\sum_{n=0}^{\infty} \alpha_n = \infty$ .

*Let  $x_0 \in D$  be any given point and  $\{x_n\}$  be the modified Ishikawa iterative scheme defined by (1.1).*

(1) *If  $\{x_n\}$  converges strongly to a fixed point  $q$  of  $T$  in  $D$ , then there exists a nondecreasing function  $\phi : [0, \infty) \rightarrow [0, \infty)$ ,  $\phi(0) = 0$ , such that*

$$\langle T^n y_n - q, J(y_n - q) \rangle \leq k_n \|y_n - q\|^2 - \phi(\|y_n - q\|)$$

*for all  $n \geq 0$ .*

(2) *Conversely, if there exists a strictly increasing function  $\phi : [0, \infty) \rightarrow [0, \infty)$ ,  $\phi(0) = 0$ , satisfying the condition (2.1), then  $x_n \rightarrow q \in F(T)$ .*

*Proof.* Since  $E$  is uniformly smooth, by Lemma 1.2, the normalized duality mapping  $J : E \rightarrow E^*$  is single-valued and uniformly continuous on any bounded subset of  $E$ .

( $\Rightarrow$ ) Let  $x_n \rightarrow q \in F(T)$ . Since  $D$  is a bounded set,  $\{T^n x_n\}$  and  $\{T^n y_n\}$  both are bounded subsets in  $D$ . Besides, since  $\beta_n \rightarrow 0$ , we have

$$y_n = (1 - \beta_n)x_n + \beta_n T^n x_n \rightarrow q \quad (n \rightarrow \infty).$$

Define

$$K = \sup_{n \geq 0} \{\|y_n - q\|\} < \infty.$$

If  $K = 0$ , then  $y_n = q$  for all  $n \geq 0$ . Hence (2.1) is true for all  $n \geq 0$ .

If  $K > 0$ , define

$$\begin{aligned} G_t &= \{n \in \mathbb{N} : \|y_n - q\| \geq t\}, & t \in (0, K), \\ G_K &= \{n \in \mathbb{N} : \|y_n - q\| = K\}, \end{aligned}$$

where  $\mathbb{N}$  is the set of all nonnegative integers. Since  $y_n \rightarrow q$ , for any  $t \in (0, K]$ , there exists  $n_0 \in \mathbb{N}$  such that, for any  $n \geq n_0$ ,

$$\|y_n - q\| < t.$$

This implies that for each  $t \in (0, K)$

- (a)  $G_t$  is a nonempty finite subset of  $\mathbb{N}$ ,
- (b)  $G_{t_1} \subset G_{t_2}$ , if  $t_1 \geq t_2$ , for all  $t_1, t_2 \in (0, K)$ ,
- (c)  $G_K = \bigcap_{t \in (0, K)} G_t$ . Since  $T : D \rightarrow D$  is asymptotically pseudo-contractive, for given  $q \in F(T)$ , we have

$$\langle T^n y_n - q, J(y_n - q) \rangle \leq k_n \|y_n - q\|^2, \quad n \geq 0.$$

By virtue of (2.2), we define a function

$$g(t) = \min_{n \in G_t} \{k_n \|y_n - q\|^2 - \langle T^n y_n - q, J(y_n - q) \rangle\}, \quad t \in (0, K).$$

From (2.2) and the property (b), we know that

- (a)  $g(t) \geq 0$  for all  $t \in (0, K)$ ,
- (b)  $g(t)$  is nondecreasing in  $t \in (0, K)$ .

Next we define a function

$$\phi(t) = \begin{cases} 0 & \text{if } t = 0, \\ g(t) & \text{if } t \in (0, K), \\ \lim_{s \rightarrow K-} g(s) & \text{if } t \in [K, \infty). \end{cases}$$

Hence  $\phi : [0, \infty) \rightarrow [0, \infty)$  is nondecreasing and  $\phi(0) = 0$ . For any  $n \geq 0$ , let  $t_n = \|y_n - q\|$ .

If  $t_n = 0$ , then  $y_n = q$  and hence  $\phi(\|y_n - q\|) = 0$ . Thus

$$\langle T^n y_n - q, J(y_n - q) \rangle = k_n \|y_n - q\|^2 - \phi(\|y_n - q\|).$$

If  $t_n \in (0, K)$ , then  $n \in G_{t_n}$  and so

$$\begin{aligned} \phi(\|y_n - q\|) &= g(t_n) \\ &= \min_{m \in G_{t_n}} \{k_m \|y_m - q\|^2 - \langle T^m y_m - q, J(y_m - q) \rangle\} \\ &\leq k_n \|y_n - q\|^2 - \langle T^n y_n - q, J(y_n - q) \rangle. \end{aligned}$$

If  $t_n = K$ , then  $n \in G_K = \bigcap_{s \in (0, K)} G_s$  and so

$$\begin{aligned} \phi(\|y_n - q\|) &= g(t_n) = \lim_{s \rightarrow K-} g(s) \\ &= \lim_{s \rightarrow K-} \min_{m \in G_s} \{k_m \|y_m - q\|^2 - \langle T^m y_m - q, J(y_m - q) \rangle\} \\ &\leq k_n \|y_n - q\|^2 - \langle T^n y_n - q, J(y_n - q) \rangle. \end{aligned}$$

Thus, the necessity is proved.

( $\Leftarrow$ ) From Lemma 1.1 and the condition (2.1), we have

$$\begin{aligned}
 & \|x_{n+1} - q\|^2 \\
 &= \|(1 - \alpha_n)(x_n - q) + \alpha_n(T^n y_n - q)\|^2 \\
 (2.3) \quad &\leq (1 - \alpha_n)^2 \|x_n - q\|^2 + 2\alpha_n \langle T^n y_n - q, J(x_{n+1} - q) \rangle \\
 &= (1 - \alpha_n)^2 \|x_n - q\|^2 + 2\alpha_n \langle T^n y_n - q, J(x_{n+1} - q) - J(y_n - q) \rangle \\
 &\quad + 2\alpha_n \langle T^n y_n - q, J(y_n - q) \rangle.
 \end{aligned}$$

Now we consider the second term on the right side of (2.3). Since  $\{T^n y_n - y_n\}$ ,  $\{x_n - T^n x_n\}$  both are bounded and

$$\begin{aligned}
 x_{n+1} - q - (y_n - q) &= (1 - \alpha_n)(x_n - y_n) + \alpha_n(T^n y_n - y_n) \\
 &= (1 - \alpha_n)\beta_n(x_n - T^n x_n) + \alpha_n(T^n y_n - y_n),
 \end{aligned}$$

we have  $x_{n+1} - q - (y_n - q) \rightarrow \theta$  ( $n \rightarrow \infty$ ). By the uniform continuity of  $J$  and the boundedness of  $\{T^n y_n - q\}$ , it follows that

$$(2.4) \quad p_n := \langle T^n y_n - q, J(x_{n+1} - q) - J(y_n - q) \rangle \rightarrow 0 \quad (n \rightarrow \infty).$$

Substituting (2.4) and (2.1) into (2.3), we have

$$\begin{aligned}
 (2.5) \quad & \|x_{n+1} - q\|^2 \leq (1 - \alpha_n)^2 \|x_n - q\|^2 + 2\alpha_n p_n \\
 & \quad + 2\alpha_n \{k_n \|y_n - q\|^2 - \phi(\|y_n - q\|)\}.
 \end{aligned}$$

Next we make an estimation for  $\|y_n - q\|^2$ .

$$\begin{aligned}
 (2.6) \quad & \|y_n - q\|^2 = \|(1 - \beta_n)(x_n - q) + \beta_n(T^n x_n - q)\|^2 \\
 & \leq (1 - \beta_n)^2 \|x_n - q\|^2 + 2\beta_n \langle T^n x_n - q, J(y_n - q) \rangle \\
 & \leq (1 - \beta_n)^2 \|x_n - q\|^2 + 2\beta_n M_1,
 \end{aligned}$$

where

$$M_1 = \sup_{n \geq 0} \{\|T^n x_n - q\| \cdot \|y_n - q\|\} < \infty.$$

Substituting (2.6) into (2.5) and using  $M_2 = \sup_{n \geq 0} \|x_n - q\|^2$  to simplify, we have

$$\begin{aligned}
 (2.7) \quad & \|x_{n+1} - q\|^2 \\
 & \leq [(1 - \alpha_n)^2 + 2\alpha_n k_n (1 - \beta_n)^2] \|x_n - q\|^2 \\
 & \quad + 2\alpha_n (p_n + 2\beta_n k_n M_1) - 2\alpha_n \phi(\|y_n - q\|) \\
 & = \|x_n - q\|^2 - \alpha_n \phi(\|y_n - q\|) - \alpha_n \{\phi(\|y_n - q\|) \\
 & \quad - [-2 + \alpha_n + 2k_n(1 - \beta_n)^2] M_2 - 2(p_n + 2\beta_n k_n M_1)\}.
 \end{aligned}$$

Let

$$\sigma = \inf_{n \geq 0} \{\|y_n - q\|\}.$$

Next we prove that  $\sigma = 0$ . Suppose the contrary, if  $\sigma > 0$ , then  $\|y_n - q\| \geq \sigma > 0$  for all  $n \geq 0$ . Hence  $\phi(\|y_n - q\|) \geq \phi(\sigma) > 0$ . From (2.7), we have

$$(2.8) \quad \begin{aligned} \|x_{n+1} - q\|^2 &\leq \|x_n - q\|^2 - \alpha_n \phi(\sigma) \\ &\quad - \alpha_n \{\phi(\sigma) - [-2 + \alpha_n + 2k_n]M_2 \\ &\quad - 2(p_n + 2\beta_n k_n M_1)\}. \end{aligned}$$

Since  $\alpha_n \rightarrow 0$ ,  $\beta_n \rightarrow 0$ ,  $p_n \rightarrow 0$  and  $k_n \rightarrow 1$ , there exists  $n_1$  such that, for all  $n \geq n_1$ ,

$$\phi(\sigma) - [-2 + \alpha_n + 2k_n]M_2 - 2(p_n + 2\beta_n k_n M_1) > 0.$$

Hence, from (2.8), it follows that

$$\|x_{n+1} - q\|^2 \leq \|x_n - q\|^2 - \alpha_n \phi(\sigma)$$

for all  $n \geq n_1$ , i.e.,

$$\alpha_n \phi(\sigma) \leq \|x_n - q\|^2 - \|x_{n+1} - q\|^2$$

for all  $n \geq n_1$ . Therefore, for any  $m \geq n_1$ , we have

$$(2.9) \quad \sum_{n=n_1}^m \alpha_n \phi(\sigma) \leq \|x_{n_1} - q\|^2 - \|x_{m+1} - q\|^2 \leq \|x_{n_1} - q\|^2.$$

Let  $m \rightarrow \infty$  in (2.9), we have

$$\infty = \sum_{n=n_1}^{\infty} \alpha_n \phi(\sigma) \leq \|x_{n_1} - q\|^2,$$

which is a contradiction and so  $\sigma = 0$ . Therefore there exists a subsequence  $\{y_{n_j}\}$  of  $\{y_n\}$  such that

$$y_{n_j} \rightarrow q \quad (n_j \rightarrow \infty),$$

i.e.,

$$y_{n_j} = (1 - \beta_{n_j})x_{n_j} + \beta_{n_j}T^{n_j}x_{n_j} \rightarrow q \quad (n_j \rightarrow \infty).$$

Since  $\alpha_n \rightarrow 0$ ,  $\beta_n \rightarrow 0$  and  $\{T^{n_j}x_{n_j}\}$ ,  $\{T^{n_j}y_{n_j}\}$  both are bounded, this implies that  $x_{n_j} \rightarrow q \quad (n_j \rightarrow \infty)$ , which implies that

$$x_{n_j+1} = (1 - \alpha_{n_j})x_{n_j} + \alpha_{n_j}T^{n_j}y_{n_j} \rightarrow q \quad (n_j \rightarrow \infty)$$

and so

$$y_{n_j+1} = (1 - \beta_{n_j+1})x_{n_j+1} + \beta_{n_j+1}T^{n_j+1}x_{n_j+1} \rightarrow q \quad (n_j \rightarrow \infty).$$

By induction, we can prove that

$$x_{n_j+i} \rightarrow q, \quad y_{n_j+i} \rightarrow q \quad (n_j \rightarrow \infty)$$

for all  $i \geq 0$ , which implies that  $x_n \rightarrow q$ . This completes the proof.

From Theorem 2.1 and Proposition 1.1, we can obtain the following theorem:

**Theorem 2.2.** *Let  $E$  be a real uniformly smooth Banach space,  $D$  be a nonempty bounded closed convex subset of  $E$  and  $T : D \rightarrow D$  be an asymptotically nonexpansive mapping with a sequence  $\{k_n\} \subset [1, \infty)$ ,  $\lim_{n \rightarrow \infty} k_n = 1$  and  $F(T) \neq \emptyset$ . Let  $\{\alpha_n\}$  and  $\{\beta_n\}$  be two real sequences in  $[0, 1]$  satisfying the conditions (i) and (ii) in Theorem 2.1. Let  $x_0 \in D$  be any given point and  $\{x_n\}$  be the modified Ishikawa iterative scheme defined by (1.1).*

(1) *If  $\{x_n\}$  converges strongly to  $q \in F(T)$ , then there exists a nondecreasing function  $\phi : [0, \infty) \rightarrow [0, \infty)$ ,  $\phi(0) = 0$ , such that*

$$(2.10) \quad \langle T^n y_n - q, J(x_n - q) \rangle \leq k_n \|y_n - q\|^2 - \phi(\|y_n - q\|)$$

for all  $n \geq 0$ .

(2) *Conversely, if there exists a strictly increasing function  $\phi : [0, \infty) \rightarrow [0, \infty)$ ,  $\phi(0) = 0$ , satisfying the condition (2.10), then  $x_n \rightarrow q \in F(T)$ .*

**Theorem 2.3.** *Let  $E$  be a uniformly convex and uniformly smooth real Banach space,  $D$  be a nonempty bounded closed convex subset of  $E$  and  $T : D \rightarrow D$  be a nonexpansive mapping. Let  $\{\alpha_n\}$  and  $\{\beta_n\}$  be two sequences in  $[0, 1]$  satisfying the conditions (i) and (ii) in Theorem 2.1. Let  $\{x_n\}$  be the modified Ishikawa iterative scheme defined by (1.1). Then  $\{x_n\}$  converges strongly to  $q \in F(T)$  if and only if there exists a strictly increasing function  $\phi : [0, \infty) \rightarrow [0, \infty)$ ,  $\phi(0) = 0$ , satisfying the following condition:*

$$(2.11) \quad \langle T^n y_n - q, J(y_n - q) \rangle \leq \|y_n - q\|^2 - \phi(\|y_n - q\|)$$

for all  $n \geq 0$ .

*Proof.* Since  $T : D \rightarrow D$  is a nonexpansive mapping, by Proposition 1.1,  $T$  is an asymptotically nonexpansive mapping with a constant sequence  $\{1\}$ , and so it is also an asymptotically pseudo-contractive mapping with the same constant sequence  $\{1\}$ . By Goebel and Kirk [5],  $F(T) \neq \emptyset$ . Therefore, the sufficiency of Theorem 2.3 follows from Theorem 2.1 immediately.

Next we prove the necessity of Theorem 2.3. Since  $E$  is a uniformly convex Banach space, the normalized duality mapping  $J : E \rightarrow E^*$  is single-valued ([4]). Let  $x_n \rightarrow q \in F(T)$ . Since  $\{T^n x_n\}$  is bounded, we know that

$$y_n = (1 - \beta_n)x_n + \beta_n T^n x_n \rightarrow q \quad (n \rightarrow \infty).$$

By the same way as given in proving necessity of Theorem 2.1, let us define

$$K = \sup_{n \geq 0} \{\|y_n - q\|\} < \infty,$$

$$G_t = \{n \in \mathbb{N} : \|y_n - q\| \geq t\}, \quad t \in (0, K),$$

$$G_K = \{n \in \mathbb{N} : \|y_n - q\| = K\},$$

$$g(t) = \min_{n \in G_t} \{\|y_n - q\|^2 - \langle T^n y_n - q, J(y_n - q) \rangle\}, \quad t \in (0, K).$$



In Theorem 2.1, we have proved that  $g(t)$  is nondecreasing and  $g(t) \geq 0$  for all  $t \in (0, K)$ .

Next we prove that  $g(t) > 0$  for all  $t \in (0, K)$ . Suppose that there exists  $t_0 \in (0, K)$  such that  $g(t_0) = 0$ . Since  $G_{t_0}$  is a finite set, there exists an  $n_0 \in G_{t_0}$  such that

$$(2.12) \quad 0 = g(t_0) = \|y_{n_0} - q\|^2 - \langle T^{n_0} y_{n_0} - q, J(y_{n_0} - q) \rangle.$$

Since  $T : D \rightarrow D$  is nonexpansive and  $q \in F(T)$ , a mapping  $T^{n_0} : D \rightarrow D$  is also nonexpansive and  $q \in F(T^{n_0})$ . By Lemma 1.3, it follows from (2.12) that  $y_{n_0} = q$ , i.e.,  $\|y_{n_0} - q\| = 0$ . However, since  $n_0 \in G_{t_0}$ , by the definition of  $G_{t_0}$ , we have  $\|y_{n_0} - q\| \geq t_0 > 0$ . This is a contradiction. Therefore  $g(t) > 0$  for all  $t \in (0, K)$ . Now we define a function

$$\phi(t) = \begin{cases} 0 & \text{if } t = 0, \\ \frac{t}{1+t} g(t) & \text{if } t \in (0, K), \\ \frac{t}{1+t} \lim_{s \rightarrow K-} g(s) & \text{if } t \in [K, \infty). \end{cases}$$

Since  $g$  is nondecreasing and  $g(t) > 0$  for all  $t \in (0, K)$ ,  $\phi : [0, \infty) \rightarrow [0, \infty)$  is strictly increasing and  $\phi(0) = 0$ . By the same way as given in the proof of Theorem 2.1, we can prove that  $\phi$  satisfies the condition (2.11). This completes the proof.

**Theorem 2.4.** *Let  $E$  be a real uniformly smooth Banach space,  $D$  be a nonempty bounded closed convex subset of  $E$  and  $T : D \rightarrow D$  be an asymptotically pseudocontractive mapping with a sequence  $\{k_n\} \subset [1, \infty)$ ,  $\lim_{n \rightarrow \infty} k_n = 1$  and  $F(T) \neq \emptyset$ . Let  $\{\alpha_n\}$  be a real sequences in  $[0, 1]$  satisfying the following conditions:*

- (i)  $\alpha_n \rightarrow 0$  ( $n \rightarrow \infty$ ),
- (ii)  $\sum_{n=0}^{\infty} \alpha_n = \infty$ .

*Let  $x_0 \in D$  be any given point and  $\{x_n\}$  be the modified Mann iterative scheme defined by (1.2).*

(1) *If  $\{x_n\}$  converges strongly to a fixed point  $q$  of  $T$ , then there exists a nondecreasing function  $\phi : [0, \infty) \rightarrow [0, \infty)$ ,  $\phi(0) = 0$ , such that*

$$(2.13) \quad \langle T^n x_n - q, J(x_n - q) \rangle \leq k_n \|x_n - q\|^2 - \phi(\|x_n - q\|)$$

*for all  $n \geq 0$ .*

(2) *Conversely, if there exists a strictly increasing function  $\phi : [0, \infty) \rightarrow [0, \infty)$ ,  $\phi(0) = 0$ , satisfying the condition (2.13), then  $x_n \rightarrow q \in F(T)$ .*

*Proof.* Taking  $\beta_n = 0$  for all  $n \geq 0$  in Theorem 2.1, then we have  $y_n = x_n$  for all  $n \geq 0$ . Therefore, the conclusion of Theorem 2.4 follows from Theorem 2.1 immediately.

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## INEQUALITIES INVOLVING MAPPINGS ASSOCIATED TO HADAMARD'S INEQUALITIES

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**ABSTRACT.** In this paper, some improvements and generalizations of the results proved by Dragomir, Cho and Kim [8] are given. Some related results in connection to Ostrowski's inequality are also given.

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*Key Words and Phrases.* Inequalities of Hadamard's and Ostrowski's type, Lipschitzian mapping, bounded variation.

### 1. Introduction and Preliminaries

Let  $I \subseteq \mathbb{R}$  be an interval of  $\mathbb{R}$  and  $f : I \rightarrow \mathbb{R}$  be a given function. If  $f$  is convex on  $I$ , then, for  $a, b \in I$  with  $a < b$ , the following inequalities are valid:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \quad (1.1)$$

These inequalities are known in the literature as Hadamard's inequalities. In connection with Hadamard's inequalities, two mappings  $H, F : [0, 1] \rightarrow \mathbb{R}$  have been considered in [3]–[5] and [8]. These mappings are defined, for any function  $f : I \rightarrow \mathbb{R}$  continuous on  $I$  and for fixed  $a, b \in I$  with  $a < b$ , as follows:

$$H(t) = \frac{1}{b-a} \int_a^b f\left(tx + (1-t)\frac{a+b}{2}\right) dx \quad (1.2)$$

and

$$F(t) = \frac{1}{(b-a)^2} \int_a^b \int_a^b f(tx + (1-t)y) dx dy \quad (1.3)$$

for all  $t \in [0, 1]$ , respectively. Under assumption that  $f$  is convex on  $I$ , the following properties of the functions  $H$  and  $F$  have been established in [4] and [5]:

- (1)  $H$  and  $F$  are convex on  $[0, 1]$ ,
- (2)  $H$  is monotonically nondecreasing on  $[0, 1]$  and  $F$  is monotonically nonincreasing on  $[0, \frac{1}{2}]$  and nondecreasing on  $[\frac{1}{2}, 1]$ ,
- (3) the following equalities are valid:

$$\begin{aligned} \inf_{t \in [0, 1]} H(t) &= H(0) = f\left(\frac{a+b}{2}\right), \\ \sup_{t \in [0, 1]} H(t) &= H(1) = \frac{1}{b-a} \int_a^b f(x) dx, \\ \inf_{t \in [0, 1]} F(t) &= F(1/2) = \frac{1}{(b-a)^2} \int_a^b \int_a^b f\left(\frac{x+y}{2}\right) dx dy, \\ \sup_{t \in [0, 1]} F(t) &= F(0) = F(1) = \frac{1}{b-a} \int_a^b f(x) dx, \end{aligned}$$

- (4) the inequality

$$F(t) \geq \max\{H(t), H(1-t)\}$$

holds for all  $t \in [0, 1]$ .

Obviously, a consequence of the above listed properties of the mappings  $H$  and  $F$  is a certain refinement of Hadamard's inequalities (1.1). On the other side, the convexity assumption on  $f$  can be weakened, for example, assuming  $f$  to be a Lipschitzian function or a function of bounded variation on  $[a, b] \subseteq I$ . In this case, some Hadamard's type inequalities involving  $H$  and  $F$  are valid.

In this paper, we are interested in such results as well as in the results related to the well known Ostrowski inequality [11, p. 468]

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left[ \frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right] (b-a)M, \quad (1.4)$$

which holds for all  $x \in [a, b]$ , provided  $f$  is  $M$ -Lipschitzian on  $I$ . We list below some recently obtained results of this type. Throughout the rest of this section, we assume that  $I \subseteq \mathbb{R}$  is an interval of  $\mathbb{R}$ ,  $f : I \rightarrow \mathbb{R}$  is a given function and  $a, b \in I$  with  $a < b$ .

**Hadamard's type inequalities involving  $H$  and  $F$ .** The following three results involving the mappings  $H$  and  $F$  have been proved in [8]:

**Theorem 1.1.** *Let the mapping  $H$  be defined by (1.2) and  $f : I \rightarrow \mathbb{R}$  be an  $M$ -Lipschitzian function on  $I$  for some constant  $M > 0$ . Then we have the following:*

- (1) *the mapping  $H$  is  $\frac{M}{4}(b-a)$ -Lipschitzian on  $[0, 1]$ ,*
- (2) *for all  $t \in [0, 1]$ , the following inequalities are valid:*

$$\left| H(t) - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{M(1-t)}{4}(b-a), \quad (1.5)$$

$$\left| f\left(\frac{a+b}{2}\right) - H(t) \right| \leq \frac{Mt}{4}(b-a) \quad (1.6)$$

and

$$\left| H(t) - t \frac{1}{b-a} \int_a^b f(x) dx - (1-t)f\left(\frac{a+b}{2}\right) \right| \leq \frac{t(1-t)M}{2}(b-a). \quad (1.7)$$

**Theorem 1.2.** *Under assumptions of Theorem 1.1, we have*

$$\left| \frac{1}{2} \left[ f\left(tb + (1-t)\frac{a+b}{2}\right) + f\left(ta + (1-t)\frac{a+b}{2}\right) \right] - H(t) \right| \leq \frac{Mt}{3}(b-a) \quad (1.8)$$

for all  $t \in [0, 1]$ .

**Theorem 1.3.** *Let the mappings  $H$  and  $F$  be defined by (1.2) and (1.3), respectively, and  $f : I \rightarrow \mathbb{R}$  be an  $M$ -Lipschitzian function on  $I$  for some constant  $M > 0$ . Then we have the following:*

- (1) *the mapping  $F$  is symmetrical with respect to  $\frac{1}{2}$ , i.e.,  $F(t) = F(1-t)$  for all  $t \in [0, 1]$ ,*
- (2) *the mapping  $F$  is  $\frac{M(b-a)}{3}$ -Lipschitzian on  $[0, 1]$ ,*
- (3) *for all  $t \in [0, 1]$ , the following inequalities are valid:*

$$\left| F(t) - \frac{1}{(b-a)^2} \int_a^b \int_a^b f\left(\frac{x+y}{2}\right) dx dy \right| \leq \frac{M(2t-1)}{6}(b-a), \quad (1.9)$$

$$\left| F(t) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{Mt}{3}(b-a) \quad (1.10)$$

and

$$|F(t) - H(t)| \leq \frac{M(1-t)}{4}(b-a). \quad (1.11)$$

**Ostrowski's type inequalities.** The following result related to Hadamard's inequalities (1.1) has been also proved in [8]:

**Theorem 1.4.** *Let  $f : I \rightarrow \mathbb{R}$  be an  $M$ -Lipschitzian function on  $I$  and let  $a, b \in I$  with  $a < b$ . Then we have*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{M}{4}(b-a) \quad (1.12)$$

and

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{M}{3}(b-a). \quad (1.13)$$

Let us note that the inequality (1.12) is a simple consequence of Ostrowski's inequality (1.4). On the other hand, the inequality (1.13) is valid with the constant  $\frac{1}{4}$  instead of the constant  $\frac{1}{3}$  on the right hand side, i.e.,

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{M}{4}(b-a). \quad (1.13')$$

The inequality (1.13') has been proved in [2] as a consequence of the following result (note that (1.13') has been also proved in [1] and [6]):

**Theorem 1.5.** *Let the assumptions of Theorem 1.4 be satisfied. Then we have*

$$\begin{aligned} & \left| \frac{(x-a)f(a) + (b-x)f(b)}{b-a} - \frac{1}{b-a} \int_a^b f(t) dt \right| \\ & \leq \left[ \frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a)M \end{aligned} \quad (1.14)$$

for all  $x \in [a, b]$ .

Further, a related result, which holds for a functions of bounded variation, has been proved in [7].

**Theorem 1.6.** Let  $f : I \rightarrow \mathbb{R}$  be of bounded variation on  $[a, b] \subseteq I$ . Then we have, for all  $x \in [a, b]$ ,

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left[ \frac{1}{2} + \frac{|x - \frac{a+b}{2}|}{b-a} \right] V_a^b(f), \quad (1.15)$$

where  $V_a^b(f) = \int_a^b |df(t)|$  denotes the total variation of  $f$  on  $[a, b]$ .

Note that the above result is also a consequence of Theorem 2 from [2]. The same is true for the following result:

**Theorem 1.7.** Let the assumptions of Theorem 1.6 be satisfied. Then we have

$$\begin{aligned} & \left| \frac{(x-a)f(a) + (b-x)f(b)}{b-a} - \frac{1}{b-a} \int_a^b f(t) dt \right| \\ & \leq \left[ \frac{1}{2} + \frac{|x - \frac{a+b}{2}|}{b-a} \right] V_a^b(f) \end{aligned} \quad (1.16)$$

for all  $x \in [a, b]$ .

Moreover, the following related result is also valid ([9], [10]):

**Theorem 1.8.** Let  $f : I \rightarrow \mathbb{R}$  be a differentiable function on  $[a, b] \subseteq I$ . If  $f' \in L_q[a, b]$  for some  $q > 1$  and  $p > 1$  is such that  $\frac{1}{p} + \frac{1}{q} = 1$ , then we have, for all  $x \in [a, b]$ ,

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left[ \frac{(x-a)^{p+1} + (b-x)^{p+1}}{(p+1)(b-a)^p} \right]^{\frac{1}{p}} \|f'\|_q, \quad (1.17)$$

where  $\|f'\|_q = \left( \int_a^b |f'(t)|^q dt \right)^{\frac{1}{q}}$  denotes the norm of  $f'$  in  $L_q[a, b]$ .

Note that the above result is also a consequence of Theorem 4 from [2] as well as the following result:

**Theorem 1.9.** Let the assumptions of Theorem 1.8 be satisfied. Then we have

$$\begin{aligned} & \left| \frac{(x-a)f(a) + (b-x)f(b)}{b-a} - \frac{1}{b-a} \int_a^b f(t) dt \right| \\ & \leq \left[ \frac{(x-a)^{p+1} + (b-x)^{p+1}}{(p+1)(b-a)^p} \right]^{\frac{1}{p}} \|f'\|_q \end{aligned} \quad (1.18)$$

for all  $x \in [a, b]$ .

In this paper, we give some improvements and generalizations of the above results.

## 2. The Main Results

As in the preceding section, assume that  $I$  is an interval of  $\mathbb{R}$  and  $a, b \in I$  with  $a < b$ . For a given function  $f : I \rightarrow \mathbb{R}$ , let the functions  $H$  and  $F$  are defined by (1.2) and (1.3), respectively. Further, for any fixed  $t \in [0, 1]$ , define

$$a_t = ta + (1-t)\frac{a+b}{2}, \quad b_t = tb + (1-t)\frac{a+b}{2}, \quad I_t = [a_t, b_t].$$

Throughout this section, we use the above notation. Note that  $I_0 = \{\frac{a+b}{2}\}$ ,  $I_1 = [a, b]$  and also

$$\frac{a_t + b_t}{2} = \frac{a+b}{2}, \quad b_t - a_t = t(b-a) \quad (2.1)$$

for all  $t \in [0, 1]$ .

Now we give our main results in this paper.

**Theorem 2.1.** *Let  $f : I \rightarrow \mathbb{R}$  be an  $M$ -Lipschitzian function on  $I_t$  for some fixed  $t$ ,  $0 < t \leq 1$ , with some constant  $M > 0$  which may depend on  $t$ . Then we have*

$$|f(x) - H(t)| \leq \left[ \frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{t^2(b-a)^2} \right] t(b-a)M \quad (2.2)$$

and

$$\begin{aligned} & \left| H(t) - t \frac{1}{b-a} \int_a^b f(s) ds - (1-t)f(x) \right| \\ & \leq \left[ \frac{1}{2} + \frac{\left(x - \frac{a+b}{2}\right)^2}{t^2(b-a)^2} \right] t(1-t)(b-a)M \end{aligned} \quad (2.3)$$

for all  $x \in I_t$ .

*Proof.* For  $0 < t \leq 1$ , we have

$$H(t) = \frac{1}{b-a} \int_a^b f\left(ts + (1-t)\frac{a+b}{2}\right) ds = \frac{1}{t(b-a)} \int_{a_t}^{b_t} f(u) du. \quad (2.4)$$

Now, applying Ostrowski's inequality (1.4) to the interval  $I_t$  and using (2.1), we have

$$|f(x) - H(t)| = \left| f(x) - \frac{1}{t(b-a)} \int_{a_t}^{b_t} f(u) du \right| \leq \left[ \frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{t^2(b-a)^2} \right] t(b-a)M,$$

which proves the inequality (2.2). The inequality (2.3) follows via the triangle inequality after adding (1.5) multiplied by  $t$  and (2.2) multiplied by  $1-t$ . This completes the proof.



**Remark 2.1.** The inequalities proved in Theorem 2.1 generalize the inequalities (1.6) and (1.7) from Theorem 1.1. Namely, setting  $x = \frac{a+b}{2}$  in (2.2) and (2.3), we get the inequalities (1.6) and (1.7), respectively.

**Theorem 2.2.** Let  $f : I \rightarrow \mathbb{R}$  be an  $M$ -Lipschitzian function on  $I_t$  for some fixed  $t$ ,  $0 \leq t \leq 1$ , with some constant  $M > 0$  which may depend on  $t$ . Then we have

$$\left| \frac{y-a}{b-a} f(a_t) + \frac{b-y}{b-a} f(b_t) - H(t) \right| \leq \left[ \frac{1}{4} + \frac{(y - \frac{a+b}{2})^2}{(b-a)^2} \right] t(b-a)M \quad (2.5)$$

for all  $y \in [a, b]$ .

*Proof.* Take any  $y \in [a, b]$ . For  $t = 0$ , the left and the right side of (2.5) are equal to zero. For  $0 < t \leq 1$ , set

$$x = ty + (1-t)\frac{a+b}{2}.$$

Then obviously  $a_t \leq x \leq b_t$  and we can apply (1.14) to the interval  $I_t$  to obtain the inequality

$$\begin{aligned} & \left| \frac{x-a_t}{b_t-a_t} f(a_t) + \frac{b_t-x}{b_t-a_t} f(b_t) - \frac{1}{b_t-a_t} \int_{a_t}^{b_t} f(u) du \right| \\ & \leq \left[ \frac{1}{4} + \frac{(x - \frac{a_t+b_t}{2})^2}{(b_t-a_t)^2} \right] (b_t-a_t)M. \end{aligned} \quad (2.6)$$

Now, using (2.1), (2.4) and

$$x - a_t = t(y - a), \quad b_t - x = t(b - y), \quad x - \frac{a_t + b_t}{2} = t\left(y - \frac{a+b}{2}\right),$$

it is easy to see that (2.6) reduces to (2.5). This completes the proof.

**Remark 2.2.** The inequality (2.5) can be regarded as an interpolation of the inequality (1.14). For  $t = 1$ , (2.5) reduces to (1.14). Further, with  $y = \frac{a+b}{2}$ , (2.5) becomes

$$\left| \frac{f(a_t) + f(b_t)}{2} - H(t) \right| \leq \frac{1}{4} t(b-a)M, \quad (2.7)$$

which is obviously an improvement of the inequality (1.8).

**Corollary 2.1.** *Let the assumptions of Theorem 2.2 be satisfied. Then we have*

$$0 \leq H(t) - f\left(\frac{a+b}{2}\right) \leq \frac{Mt}{4}(b-a) \quad (2.8)$$

and

$$0 \leq \frac{f(a_t) + f(b_t)}{2} - H(t) \leq \frac{Mt}{4}(b-a) \quad (2.9)$$

for all  $t \in [0, 1]$ .

*Proof.* The left-hand side inequalities in (2.8) and (2.9) have been proved in [8], while the right hand side inequalities in (2.8) and (2.9) follow from (2.2) with  $x = \frac{a+b}{2}$  and from (2.7), respectively.

**Remark 2.3.** If  $f$  is a differentiable and convex mapping on  $I$  such that

$$\sup_{x \in I} |f'(x)| < \infty,$$

then the inequalities (2.8) and (2.9) hold with

$$M = \sup_{x \in I_t} |f'(x)|. \quad (2.10)$$

Note that, in this case, the right hand side inequality in (2.9) is an improvement of the related result from [8] (the inequality (3.8)) in which  $\frac{1}{3}$  stands in place of  $\frac{1}{4}$ .

**Theorem 2.3.** *Let  $f : I \rightarrow \mathbb{R}$  be an  $M$ -Lipschitzian function on  $I$  for some constant  $M > 0$ . Then, for all  $t \in [0, 1]$ , we have*

$$\left| F(t) - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{M(b-a)}{3} \min\{t, 1-t\}. \quad (2.11)$$

If  $f$  is also convex on  $I$ , then

$$0 \leq \frac{1}{b-a} \int_a^b f(x) dx - F(t) \leq \frac{M(b-a)}{3} \min\{t, 1-t\} \quad (2.12)$$

for all  $t \in [0, 1]$ .

*Proof.* The inequality (2.11) is a simple consequence of the inequality (1.10), since  $F$  is symmetrical with respect to  $\frac{1}{2}$ . The left-hand side inequality in (2.12) has been proved in [8], while the right-hand side inequality in (2.12) follows from (2.11).

**Remark 2.4.** The inequality (2.11) is an improvement of the related result from [4] where  $t$  stands in place of  $\min\{t, 1 - t\}$ . The same is true for the right-hand side inequality in (2.12) for the case when  $f$  is differentiable and  $M$  is given like in (2.10) by  $M = \sup_{x \in I} |f'(x)|$ .

**Theorem 2.4.** Let  $f : I \rightarrow \mathbb{R}$  be a given function. Assume that  $f$  is of bounded variation on  $[a, b] \subseteq I$ . Then we have the following:

(1) If  $x \in I_t$  for some  $t$ ,  $0 < t \leq 1$ , then we have

$$\begin{aligned} |f(x) - H(t)| &\leq \left[ \frac{1}{2} + \frac{|x - \frac{a+b}{2}|}{t(b-a)} \right] V_{a_t}^{b_t}(f) \\ &\leq \left[ \frac{1}{2} + \frac{|x - \frac{a+b}{2}|}{t(b-a)} \right] V_a^b(f). \end{aligned} \quad (2.13)$$

(2) For a given  $y \in [a, b]$ , we have

$$\begin{aligned} \left| \frac{y-a}{b-a} f(a_t) + \frac{b-y}{b-a} f(b_t) - H(t) \right| &\leq \left[ \frac{1}{2} + \frac{|y - \frac{a+b}{2}|}{b-a} \right] V_{a_t}^{b_t}(f) \\ &\leq \left[ \frac{1}{2} + \frac{|y - \frac{a+b}{2}|}{b-a} \right] V_a^b(f) \end{aligned} \quad (2.14)$$

for all  $t \in [0, 1]$ .

*Proof.* First we use (2.4) and apply the inequality (1.15) to the interval  $I_t$  to obtain

$$|f(x) - H(t)| = \left| f(x) - \frac{1}{t(b-a)} \int_{a_t}^{b_t} f(u) du \right| \leq \left[ \frac{1}{2} + \frac{|x - \frac{a+b}{2}|}{t(b-a)} \right] V_{a_t}^{b_t}(f),$$

which is just the first inequality in (2.13). To prove the first inequality in (2.14), we argue similarly as in the proof of Theorem 2.2 using (1.16) instead of (1.14). The second inequality in (1.13) and the second inequality in (1.14) follow immediately since we have

$$V_{a_t}^{b_t}(f) \leq V_a^b(f).$$

**Remark 2.5.** The inequalities (2.13) and (2.14) interpolate the inequalities (1.15) and (1.16), respectively, and, for  $t = 1$ , reduce to them.

**Corollary 2.2.** *Let  $f : I \rightarrow \mathbb{R}$  be a given function. Assume that  $f$  is convex on  $[a, b] \subseteq I$ . Then we have*

$$0 \leq H(t) - f\left(\frac{a+b}{2}\right) \leq \frac{1}{2}V_{a_t}^{b_t}(f) \leq \frac{1}{2}V_a^b(f) \quad (2.15)$$

and

$$0 \leq \frac{f(a_t) + f(b_t)}{2} - H(t) \leq \frac{1}{2}V_{a_t}^{b_t}(f) \leq \frac{1}{2}V_a^b(f). \quad (2.16)$$

*Proof.* The left-hand side inequalities in (2.15) and (2.16) coincide with the left-hand side inequalities in (2.8) and (2.9), respectively. The right-hand side inequalities in (2.15) and (2.16) follow from (2.13) with  $x = \frac{a+b}{2}$  and (2.14) with  $y = \frac{a+b}{2}$ , respectively.

**Theorem 2.5.** *Let  $f : I \rightarrow \mathbb{R}$  be a differentiable function on  $[a, b] \subseteq I$ . Assume that  $f' \in L_q[a, b]$  for some  $q > 1$  and  $p > 1$  is such that  $\frac{1}{p} + \frac{1}{q} = 1$ . By  $\|f'\|_{q, I_t}$  denote a  $q$ -norm of the function  $f'$  on the interval  $I_t$ .*

(1) *If  $x \in I_t$  for some  $t$ ,  $0 < t \leq 1$ , then we have*

$$\begin{aligned} |f(x) - H(t)| &\leq \left[ \frac{(x - a_t)^{p+1} + (b_t - x)^{p+1}}{(p+1)t^p(b-a)^p} \right]^{\frac{1}{p}} \|f'\|_{q, I_t} \\ &\leq \left[ \frac{(x - a_t)^{p+1} + (b_t - x)^{p+1}}{(p+1)t^p(b-a)^p} \right]^{\frac{1}{p}} \|f'\|_q. \end{aligned} \quad (2.17)$$

(2) *For any given  $y \in [a, b]$ , we have*

$$\begin{aligned} &\left| \frac{y-a}{b-a}f(a_t) + \frac{b-y}{b-a}f(b_t) - H(t) \right| \\ &\leq \left[ \frac{(y-a)^{p+1} + (b-y)^{p+1}}{(p+1)(b-a)^p} \right]^{\frac{1}{p}} t^{\frac{1}{p}} \|f'\|_{q, I_t} \\ &\leq \left[ \frac{(y-a)^{p+1} + (b-y)^{p+1}}{(p+1)(b-a)^p} \right]^{\frac{1}{p}} t^{\frac{1}{p}} \|f'\|_q \end{aligned} \quad (2.18)$$

for all  $t \in [0, 1]$ .

*Proof.* To get (2.17) and (2.18), we argue analogously as in the case of (2.13) and (2.14), but using (1.17) and (1.18) instead of (1.15) and (1.16), respectively.

**Remark 2.6.** The inequalities (2.17) and (2.18) interpolate the inequalities (1.17) and (1.18), respectively, and, for  $t = 1$ , reduce to them.

**Corollary 2.3.** *Let the assumptions of Theorem 2.5 be satisfied. If  $f$  is a convex function on  $[a, b]$ , then we have*

$$0 \leq H(t) - f\left(\frac{a+b}{2}\right) \leq \frac{1}{2} \left[ \frac{t(b-a)}{p+1} \right]^{\frac{1}{p}} \|f'\|_{q, I_t} \leq \frac{1}{2} \left[ \frac{t(b-a)}{p+1} \right]^{\frac{1}{p}} \|f'\|_q \quad (2.19)$$

and

$$0 \leq \frac{f(a_t) + f(b_t)}{2} - H(t) \leq \frac{1}{2} \left[ \frac{t(b-a)}{p+1} \right]^{\frac{1}{p}} \|f'\|_{q, I_t} \leq \frac{1}{2} \left[ \frac{t(b-a)}{p+1} \right]^{\frac{1}{p}} \|f'\|_q. \quad (2.20)$$

*Proof.* The left-hand side inequalities in (2.19) and (2.20) coincide with the left-hand side inequalities in (2.8) and (2.9), respectively. The right-hand side inequalities in (2.19) and (2.20) follow from (2.17) with  $x = \frac{a+b}{2}$  and (2.18) with  $y = \frac{a+b}{2}$ , respectively.

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# A Bivariate Shape-Preserving Quasi-Interpolant Method with Positive Compactly Supported Bases

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## Abstract

Using the approach presented in [5] for the univariate case, in this paper we present a strategy to construct a bivariate quasi-interpolant shape-preserving and  $\mathbb{P}_1$ -reproducing scheme based on positive compactly supported basis functions. The proposed quasi-interpolant method linearly combines such bases with coefficients simply given by the data.

**Keywords:** Shape-preserving, Quasi-interpolation, Polynomial reproduction, Compact support bases.

## §1. Introduction

It is well known that shape-preserving interpolation and/or approximation methods are of great importance for reconstructing curves and surfaces following the shape suggested by the data. Nowadays, regarding shape-preserving interpolant methods, numerous effective methods are available using different approaches in both the one dimensional and the bidimensional case (e.g. [2], [3], [6], [7], [8] and references quoted therein). On the other hand, little is known about shape-preserving quasi-interpolant methods (see for instance [4], [5], [9] and references therein). Dealing with univariate functional data, the problem of defining an efficient quasi-interpolant strategy was confronted in [4] and in [5] where the authors basically exploited the properties of the reproducing kernel of the Hilbert space of all the functions with second derivatives in  $L^2(\mathbb{R})$ . Unfortunately, this approach cannot be extended

to the bidimensional case working with the reproducing kernel of the Hilbert space  $L^2(\mathbb{R}^2)$ . Nevertheless, the analytical structure of the univariate quasi-interpolant function defined in [5] can be generalized to the bivariate setting via a tensor product approach. Indeed, in this work we do take into account linear combinations of particular basis functions in order to get the shape-preserving quasi-interpolant looked for. The so obtained quasi-interpolant is also shown to be linear reproducing.

More precisely, given a set of structured data values  $f_{i,j}$ ,  $i = 1, \dots, n, j = 1, \dots, m$ , arranged over a rectangular grid, in this paper a strategy is presented to define a quasi-interpolant function expressed as

$$\sigma(x, y) = \sum_{i=1}^n \sum_{j=1}^m f_{i,j} B_{ij}(x, y), \quad (x, y) \in [x_1, x_n] \times [y_1, y_m] \quad (1)$$

where  $B_{ij}$  are positive compactly supported basis functions. The function  $\sigma$  is shown to be monotonicity and convexity preserving along the grid lines  $x = x_i$ ,  $x_i < x_{i+1}$ ,  $i = 1, \dots, n$  and  $y = y_j$ ,  $y_j < y_{j+1}$ ,  $j = 1, \dots, m$  and linear reproducing.

We recall that, defining the quantities  $h_i^x = x_{i+1} - x_i$ ,  $\Delta_{ij}^x f = \frac{(f_{i+1,j} - f_{i,j})}{h_i^x}$ ,  $j = 1, \dots, m$ ,  $i = 1, \dots, n-1$  ( $h_j^y = y_{j+1} - y_j$ ,  $\Delta_{ij}^y f = \frac{(f_{i,j+1} - f_{i,j})}{h_j^y}$ ,  $i = 1, \dots, n$ ,  $j = 1, \dots, m-1$ ), the data are monotone increasing along the grid lines  $y = y_j$ ,  $j = 1, \dots, m$  (along the grid lines  $x = x_i$ ,  $i = 1, \dots, n$ ) if

$$\begin{aligned} \Delta_{ij}^x f &\geq 0, \quad i = 1, \dots, n-1, j = 1, \dots, m \\ (\Delta_{ij}^y f &\geq 0, \quad i = 1, \dots, n, j = 1, \dots, m-1) \end{aligned} \quad (2)$$

and they are convex along the grid lines  $y = y_j$ ,  $j = 1, \dots, m$  (along the grid lines  $x = x_i$ ,  $i = 1, \dots, n$ ) if

$$\begin{aligned} \Delta_{ij}^x f &\leq \Delta_{i+1,j}^x f, \quad i = 1, \dots, n-2, \quad j = 1, \dots, m-1 \\ (\Delta_{ij}^y f &\leq \Delta_{i,j+1}^y f, \quad j = 1, \dots, m-2, \quad i = 1, \dots, n-1). \end{aligned} \quad (3)$$

An analogous definition is given for the monotone decreasing case and the concave case.

Concerning the linear reproduction we must require that, in case for all  $i = 1, \dots, n$ ,  $j = 1, \dots, m-1$ ,  $f_{i,j} = p(x_i, y_j)$  with  $p \in \mathbb{P}_1$  (the space of polynomials of degree one) it holds



$$\sigma(x, y) \equiv p(x, y), \quad (x, y) \in [x_1, x_n] \times [y_1, y_m]. \quad (4)$$

The paper is divided into the following sections. In section 2 the choice of the bases is discussed and a characterization of the quasi-interpolant is given. In section 3 the shape-preserving properties of the so constructed quasi-interpolant function are investigated and its linear reproduction is proved. Moreover, using the  $\mathbb{P}_1$  reproduction, the approximation order is investigated. Finally, in section 4 some graphs are shown to illustrate the performance of the proposed shape-preserving quasi-interpolant method.

## §2. Bell-Shaped Compactly Supported Bases

The aim of this section is the construction of suitable compactly supported basis functions and the investigation of their properties. The approach we are going to use is a generalization of the one presented first in [4] and then modified in [5] since we are dealing with a suitable linear combination of translates of functions of the type

$$\phi(x, y) := v(x)v(y) \quad (5)$$

$v$  being the real valued function  $v(x) := -\frac{1}{4}x|x|$ .

Given a set of abscissæ  $\{x_i\}_{i=-2, \dots, n+3}$ ,  $x_i < x_{i+1}$ ,  $i = -2, \dots, n+2$  first we denote by  $D_{2,i}^{1,x}$  the approximation of the first derivative of a given function at the point  $x_i$   $\mathbb{P}_2$ -exact, i.e. exact for every polynomial of degree two. More precisely, for any function  $g$ ,  $D_{2,i}^{1,x}g$  is defined as

$$D_{2,i}^{1,x}g(\cdot) := \sum_{k=-1}^1 \gamma_k^{i,x} g(x_{i+k}), \quad i = -1, \dots, n+1,$$

where the coefficients  $\{\gamma_k^{i,x}\}_{k=-1,0,1}$  are given by

$$\gamma_{-1}^{i,x} = \frac{-h_i^x}{h_{i-1}^x(h_{i-1}^x + h_i^x)}, \quad \gamma_0^{i,x} = \frac{h_i^x - h_{i-1}^x}{h_{i-1}^x h_i^x}, \quad \gamma_1^{i,x} = \frac{h_{i-1}^x}{h_i^x(h_{i-1}^x + h_i^x)}, \quad (6)$$

with  $h_i^x = x_{i+1} - x_i$  for all  $i = -2, \dots, n+2$  and it is such that if  $g \in \mathbb{P}_2$  then  $D_{2,i}^{1,x}g(x) = g'(x_i)$ .

Then we denote by  $C_i^x$  the univariate real valued functions

$$C_i^x(x) = \frac{1}{h_{i-1}^x} D_{2,i-1}^{1,x} v(x - \cdot) - \frac{(h_i^x + h_{i-1}^x)}{h_i^x h_{i-1}^x} D_{2,i}^{1,x} v(x - \cdot) \\ + \frac{1}{h_i^x} D_{2,i+1}^{1,x} v(x - \cdot), \quad i = 0, \dots, n+1.$$

Defining analogously the  $C_j^y$  univariate functions using  $\{y_j\}_{j=-2, \dots, m+3}$ ,  $y_j < y_{j+1}$ ,  $j = -2, \dots, m+2$ ,  $h_j^y = y_{j+1} - y_j$ ,  $j = -2, \dots, m+2$ ,  $\{\gamma_k^{j,y}\}_{k=-1,0,1}$  and  $D_{2,j}^{1,y} v(\cdot - y)$ , we choose the family of bivariate bases  $B_{ij}$  as

$$B_{ij}(x, y) := C_i^x(x) C_j^y(y), \quad i = 0, \dots, n+1, \quad j = 0, \dots, m+1. \quad (7)$$

**Remark 1.** It should be noted that  $B_{ij}$  are linear combinations of translates of the function  $\phi$  defined in (5). In fact, using the linear operator  $L_j^y$

$$L_j^y D_{2,j}^{1,y} \phi(x, y - \cdot) = \frac{1}{h_{j-1}^y} D_{2,j-1}^{1,y} \phi(x, y - \cdot) - \frac{h_j^y + h_{j-1}^y}{h_j^y h_{j-1}^y} D_{2,j}^{1,y} \phi(x, y - \cdot) + \\ \frac{1}{h_j^y} D_{2,j+1}^{1,y} \phi(x, y - \cdot)$$

and the analogous linear operator  $L_i^x$  acting on the  $x$  variable, we obtain

$$B_{ij}(x, y) = L_j^y D_{2,j}^{1,y} L_i^x D_{2,i}^{1,x} \phi(x - \cdot, y - \cdot).$$

The following theorems can be proved directly from (7) considering that every function  $C_i^x$ ,  $i = 0, \dots, n+1$  with compact support on  $[x_{i-2}, x_{i+2}]$  and every function  $C_j^y$ ,  $j = 0, \dots, m+1$  with compact support on  $[y_{j-2}, y_{j+2}]$  are positive functions and that these functions sum up 1 that is to say  $\sum_{i=0}^{n+1} C_i^x(x) = 1$ ,  $x \in [x_1, x_n]$  and  $\sum_{j=0}^{m+1} C_j^y(y) = 1$ ,  $y \in [y_1, y_m]$  (we refer the reader to [5] for all the details).

**Theorem 1.** *The bases given in (7) satisfy*

$$B_{ij}(x, y) > 0, \quad \text{for } (x, y) \in [x_{i-2}, x_{i+2}] \times [y_{j-2}, y_{j+2}], \\ B_{ij}(x, y) = 0, \quad \text{otherwise} \quad (8)$$

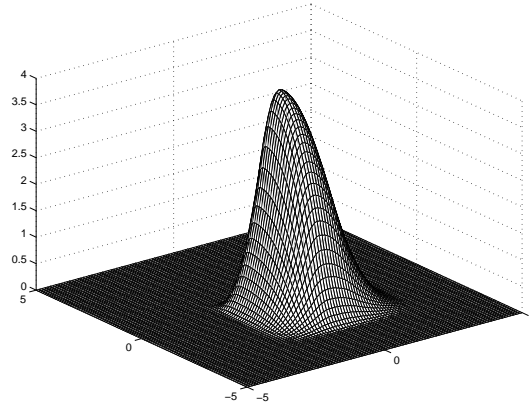
for  $i = 0, \dots, n+1$ ,  $j = 0, \dots, m+1$ .

It is worthwhile to note that the local support of the bases guarantees a locality in the shape-preservation.

**Theorem 2.** *The bases given in (7) are a partition of unity i.e.,*

$$\sum_{i=0}^{n+1} \sum_{j=0}^{m+1} B_{ij}(x, y) = 1, \quad (x, y) \in [x_1, x_n] \times [y_1, y_n] . \quad (9) \quad \blacksquare$$

Figure 1 displays the graph of a  $B_{ij}$  function with non uniform knots.



**Fig. 1.** *Graph of a  $B_{ij}$  function with non uniform knots*

We are now in a position to define the desired quasi-interpolant based on the data set  $\{x_i, y_j, f_{i,j}\}_{i=1,\dots,n, j=1,\dots,m}$  i.e.,

$$\sigma(x, y) := \sum_{i=0}^{n+1} \sum_{j=0}^{m+1} f_{i,j} B_{ij}(x, y) , \quad (10)$$

having set  $x_{1-l} = x_1 - l \cdot h_1^x$ ,  $x_{n+l} = x_n + l \cdot h_{n-1}^x$ ,  $l = 1, 2, 3$  and  $y_{1-l} = y_1 - l \cdot h_1^y$ ,  $y_{m+l} = y_m + l \cdot h_{m-1}^y$ ,  $l = 1, 2, 3$  and having set the extra values  $f_{0,j}$ ,  $f_{n+1,j}$ ,  $j = 0, \dots, m+1$ ,  $f_{i,0}$ ,  $f_{i,m+1}$ ,  $i = 0, \dots, n+1$  according to the shape of the data as explained here in the sequel. In particular, as a distinction must be made according to the shape (2) of the data, we first discuss the case of monotone increasing data along both the  $x$  and the  $y$  axes. In this case the following algorithm is proposed.

### Algorithm

```

Set  $f_{0,1} := 2f_{1,1} - f_{2,1}$     set  $f_{n+1,1} := 2f_{n,1} - f_{n-1,1}$ ;
for  $j = 2, \dots, m$ 
  if  $2f_{1,j} - f_{2,j} \geq f_{0,j-1}$ 
    set  $f_{0,j} := 2f_{1,j} - f_{2,j}$  ;
  else
    set  $f_{0,j} := f_{0,j-1}$  ;
  end
  if  $2f_{n,j} - f_{n-1,j} \geq f_{n+1,j-1}$ 
    set  $f_{n+1,j} := 2f_{n,j} - f_{n-1,j}$  ;
  else
    set  $f_{n+1,j} := f_{n+1,j-1}$  ;
  end
end
set  $f_{1,0} := 2f_{1,1} - f_{1,2}$     set  $f_{1,m+1} := 2f_{1,m} - f_{1,m-1}$ ;
for  $i = 2, \dots, n$ 
  if  $2f_{i,1} - f_{i,2} \geq f_{i-1,0}$ 
    set  $f_{i,0} := 2f_{i,1} - f_{i,2}$  ;
  else
    set  $f_{i,0} := f_{i-1,0}$  ;
  end
  if  $2f_{i,m} - f_{i,m-1} \geq f_{i-1,m+1}$ 
    set  $f_{i,m+1} := 2f_{i,m} - f_{i,m-1}$  ;
  else
    set  $f_{i,m+1} := f_{i-1,m+1}$  ;
  end
end
end

```

The corner values  $f_{0,0}$ ,  $f_{0,m+1}$ ,  $f_{n+1,0}$ ,  $f_{n+1,m+1}$  are set as the values of the linear polynomials interpolating the 3 neighboring boundary points at  $(x_0, y_0)$ ,  $(x_0, y_{m+1})$ ,  $(x_{n+1}, y_0)$ ,  $(x_{n+1}, y_{m+1})$ , respectively. For example,  $f_{0,0} = p_1(x, y)|_{(x_0, y_0)}$  with  $p_1$  being the linear polynomial through the points

$(x_1, y_0, f_{1,0}), (x_0, y_1, f_{0,1}), (x_1, y_1, f_{1,1})$ . For simplicity, we do not report here the algorithms related to monotone decreasing data in one or in both directions. We do point out that in the "if" cycles the symbol " $\leq$ " should be taken into account instead of " $\geq$ " whenever we deal with decreasing data. It is easy to show that this algorithm and the analogous ones produces extra values which agree with the shape of the data and they are " $\mathbb{P}_1$ -reproducing" meaning that, in case  $f_{i,j}$ ,  $i = 1, \dots, n$ ,  $j = 1, \dots, m$  lie on a plane, so do the added boundary values. This is a crucial point for the polynomial reproduction of  $\sigma$ .

Before concluding this section, it is worthwhile to provide a Lemma for expressing each  $C_i^x$  basis in terms of the quadratic B-splines. The proof is straightforward when we write any quadratic B-spline with knots  $\{x_{i-1}, x_i, x_{i+1}, x_{i+2}\}$ , say  $N_{2,i}$ , as

$$N_{2,i} := \frac{-2}{h_{i-1}^x(h_{i-1}^x + h_i^x)}v(x - x_{i-1}) + \frac{2(h_{i-1} + h_i + h_{i+1})}{h_{i-1}^x h_i(h_i^x + h_{i+1}^x)}v(x - x_i) \\ - \frac{2(h_{i-1} + h_i + h_{i+1})}{h_{i+1}^x h_i(h_i^x + h_{i-1}^x)}v(x - x_{i+1}) + \frac{2}{h_{i+1}^x(h_{i+1}^x + h_i^x)}v(x - x_{i+2}),$$

$v$  being the real valued function  $v(x) = -\frac{1}{4}x|x|$ .

**Lemma 1.** Any function  $C_i^x$ ,  $i = 0, \dots, n+1$  can be written as

$$C_i^x(x) = \frac{1}{2}(N_{2,i-1}(x) + N_{2,i}(x)), \quad i = 0, \dots, n+1,$$

where  $N_{2,i}$  denotes the quadratic B-spline with knots  $\{x_{i-1}, x_i, x_{i+1}, x_{i+2}\}$ .

### §3. Shape-preserving Properties and Linear Reproduction

In order to check the shape-preserving properties of  $\sigma$  and its linear reproduction, we strongly rely on the structure of the bases involved in the definition of  $\sigma$ . Doing so, we can prove the theorem here below.

**Theorem 3.** The quasi-interpolant  $\sigma$  defined in (10) with extra values defined by means of the algorithm of section 2 is monotonicity and convexity/concavity preserving.

**Proof:** First, taking into account the monotonicity preservation along the  $x$  axis, it is convenient to express  $\sigma$  as

$$\begin{aligned} \sigma(x, y) = & \sum_{i=1}^{n+1} \sum_{j=0}^{m+1} \frac{(f_{i,j} - f_{i-1,j})}{h_{i-1}^x} \tilde{D}_i(x) C_j^y(y) - \sum_{j=0}^{m+1} \frac{f_{n+1,j}}{h_{n+1}^x} \tilde{D}_{n+2}(x) C_j^y(y) \\ & + \sum_{j=0}^{m+1} \frac{f_{0,j}}{h_{-1}^x} \tilde{D}_0(x) C_j^y(y) , \end{aligned} \quad (11)$$

where  $\tilde{D}_i, i = 0, \dots, n+2$  are the univariate functions

$$\tilde{D}_i(x) = D_{2,i-1}^{1,x} v(x - x_{i-1}) - D_{2,i}^{1,x} v(x - x_i) .$$

Now, computing the first derivative of  $\sigma$  with respect to  $x$ , we get

$$\frac{\partial \sigma(x, y)}{\partial x} = \sum_{i=1}^{n+1} \sum_{j=0}^{m+1} \frac{(f_{i,j} - f_{i-1,j})}{h_{i-1}^x} \tilde{D}_i'(x) C_j^y(y) ,$$

since it is easy to prove that

$$\tilde{D}_0(x) = \frac{x_0 - x_{-1}}{2}, \quad \tilde{D}_{n+2}(x) = \frac{x_{n+1} - x_{n+2}}{2} ,$$

being  $D_{2,k}^{1,x} v(x - \cdot) = \frac{x - x_k}{2}$ ,  $k = -1, 0$ ,  $D_{2,l}^{1,x} v(x - \cdot) = -\frac{x - x_l}{2}$ ,  $l = n+1, n+2$ . The monotonicity is then stated by recalling the positivity of any  $C_j^y(y)$  and of any  $\tilde{D}_i'(x)$  (again we refer the reader to [5] for details).

Analogously, along the  $y$  axis we arrive at

$$\frac{\partial \sigma(x, y)}{\partial y} = \sum_{j=1}^{m+1} \sum_{i=0}^{n+1} \frac{(f_{i,j} - f_{i,j-1})}{h_{j-1}^y} C_i^x(x) \tilde{D}_j'(y) ,$$

thus concluding the proof of the monotonicity preservation.

With respect to the convexity/concavity preservation along the  $x$  axis, we intend to show that  $\frac{\partial \sigma(x, y)}{\partial x}$  is a monotone function. For this purpose, we use the following expression for  $\frac{\partial \sigma(x, y)}{\partial x}$

$$\begin{aligned} \frac{\partial \sigma(x, y)}{\partial x} = & - \sum_{i=1}^n \sum_{j=0}^{m+1} \frac{\Delta_{i+1j}^x f - \Delta_{ij}^x f}{2} D_{2,i}^{1,x} |x - \cdot| C_j^y(y) \\ & + \sum_{j=0}^{m+1} \frac{\Delta_{1j}^x f}{2} C_j^y(y) - \sum_{j=0}^{m+1} \frac{\Delta_{n+1j}^x f}{2} C_j^y(y) . \end{aligned}$$

Then, let  $x \in [x_i, x_{i+1}]$  for some  $i \in \{1, \dots, n-1\}$ . At each point  $x \in (x_i, x_{i+1})$  the function  $\frac{\partial \sigma(x, y)}{\partial x}$  is differentiable with respect to  $x$ , so we can write

$$\frac{\partial^2 \sigma(x, y)}{\partial x^2} = \sum_{j=0}^{m+1} (\Delta_{i+1j}^x f - \Delta_{ij}^x f) C_j^y(y) \cdot \gamma_1^{i,x} - (\Delta_{i+2j}^x f - \Delta_{i+1j}^x f) C_j^y(y) \cdot \gamma_{-1}^{i+1,x},$$

where the coefficients  $\gamma_{-1}^{i+1,x}$ ,  $\gamma_1^{i,x}$  are given in (6).

This quantity has a constant sign depending on the sign of  $(\Delta_{i+1j}^x f - \Delta_{ij}^x f)$ ,  $j = 0, \dots, m+1$  and of  $(\Delta_{i+2j}^x f - \Delta_{i+1j}^x f)$ ,  $j = 0, \dots, m+1$ . Thus,  $\frac{\partial \sigma(x, y)}{\partial x}$  is monotone in  $(x_i, x_{i+1})$ . To conclude the proof let us analyze the quantities  $\frac{\partial \sigma(x_i, y)}{\partial x} - \frac{\partial \sigma(x, y)}{\partial x}$ ,  $\frac{\partial \sigma(x, y)}{\partial x} - \frac{\partial \sigma(x_{i+1}, y)}{\partial x}$ .

After a little algebra, we obtain

$$\begin{aligned} \frac{\partial \sigma(x_i, y)}{\partial x} - \frac{\partial \sigma(x, y)}{\partial x} &= \sum_{j=0}^{m+1} (\Delta_{i+1j}^x f - \Delta_{ij}^x f) \cdot \gamma_1^{i,x} (x_i - x) C_j^y(y) \\ &\quad + \sum_{j=0}^{m+1} (\Delta_{i+2j}^x f - \Delta_{i+1j}^x f) \cdot \gamma_{-1}^{i+1,x} (x - x_i) C_j^y(y), \end{aligned}$$

$$\begin{aligned} \frac{\partial \sigma(x, y)}{\partial x} - \frac{\partial \sigma(x_{i+1}, y)}{\partial x} &= \sum_{j=0}^{m+1} (\Delta_{i+1j}^x f - \Delta_{ij}^x f) \cdot \gamma_1^{i,x} (x - x_{i+1}) C_j^y(y) \\ &\quad + \sum_{j=0}^{m+1} (\Delta_{i+2j}^x f - \Delta_{i+1j}^x f) \cdot \gamma_{-1}^{i+1,x} (x_{i+1} - x) C_j^y(y). \end{aligned}$$

The constant sign of the previous quantities concludes the proof of the convexity preservation along the  $x$  axis. Similarly, we can prove the convexity/concavity preservation along the  $y$  axis. ■

Concerning the linear reproduction of  $\sigma$ , we take into account the reproduction of the monomials  $x$  and  $y$  separately (the constant reproduction is in fact given by (9)).

**Theorem 4.** *The quasi-interpolant  $\sigma$  defined in (10) with the extra values given by the algorithm in section 2 is  $\mathbb{P}_1$  reproducing.*

**Proof:** First let us consider  $f_{i,j} = x_i$ ,  $i = 1, \dots, n$ ,  $j = 1, \dots, m$ . By (10) and by the definition of the extra values in the algorithm in section 2, we have  $\Delta_{ij}^x f = 1$  for all  $i$  and  $j$ . Thus, it holds

$$\begin{aligned}
 \sigma(x, y) &= \sum_{i=1}^n \sum_{j=0}^{m+1} (\Delta_{ij}^x f - \Delta_{i-1,j}^x f) D_{2,i}^{1,x} v(x - \cdot) C_j^y(y) \\
 &+ \sum_{j=0}^{m+1} \frac{f_{0,j}}{h_{-1}} D_{2,-1}^{1,x} v(x - \cdot) C_j^y(y) + \sum_{j=0}^{m+1} \frac{f_{1,j}}{h_0} D_{2,0}^{1,x} v(x - \cdot) C_j^y(y) \\
 &- \sum_{j=0}^{m+1} \frac{f_{0,j}(h_{-1} + h_0)}{h_{-1}h_0} D_{2,0}^{1,x} v(x - \cdot) C_j^y(y) \\
 &- \sum_{j=0}^{m+1} \frac{f_{n+1,j}(h_n + h_{n+1})}{h_n h_{n+1}} D_{2,n+1}^{1,x} v(x - \cdot) C_j^y(y) \\
 &+ \sum_{j=0}^{m+1} \frac{f_{n,j}}{h_n} D_{2,n+1}^{1,x} v(x - \cdot) C_j^y(y) \\
 &+ \sum_{j=0}^{m+1} \frac{f_{n+1,j}}{h_{n+1}} D_{2,n+2}^{1,x} v(x - \cdot) C_j^y(y) \\
 &= \frac{x}{2} + \frac{x}{2} = x,
 \end{aligned}$$

where we used (11) and  $\sum_{j=0}^{m+1} C_j^y(y) = 1$  (as proved in [5]). The same holds for the reproduction of  $y$ , thus concluding the proof. ■

In the following, using the just proved  $\mathbb{P}_1$  reproduction of the quasi interpolant, its approximation order is also investigated.

**Theorem 5.** *Let  $R := [x_1, x_n] \times [y_1, y_m]$  and let  $f \in C^2(R)$ . Furthermore, let us denote by  $R_{i,j} := [x_i, x_{i+1}] \times [y_j, y_{j+1}]$  the generic sub-rectangle of the grid defined over  $R$ . Then, there is a constant  $\mathcal{K}$ , depending on  $\|\frac{\partial^2 f(x,y)}{\partial x^2}\|_\infty$ ,  $\|\frac{\partial^2 f(x,y)}{\partial xy}\|_\infty$  and  $\|\frac{\partial^2 f(x,y)}{\partial y^2}\|_\infty$  such that setting  $\Omega_{i,j} := \max\{h_{i-1}^x, h_{j-1}^y, h_i^x + h_{i+1}^x, h_j^y + h_{j+1}^y\}$  it holds*

$$\|f - \sigma\|_{\infty, R_{i,j}} \leq \mathcal{K} \Omega_{i,j}^2 \quad (12)$$



where  $\|\cdot\|_{\infty, R_{i,j}}$  is the infinity norm on  $R_{i,j}$ .

**Proof:** The proof is mainly based on the  $\mathbb{P}_1$  reproduction of the quasi interpolant  $\sigma$  already proved. In fact, considering the support of the  $C_i^x$  and  $C_j^y$  function, on the rectangle  $R_{i,j}$  the quasi-interpolant  $\sigma$  reads as

$$\sigma(x, y) = \sum_{k=i-1}^{i+2} \sum_{l=j-1}^{j+2} f_{k,l} C_k^x(x) C_l^y(y).$$

Now, using a Taylor expansion of the function  $f$  around the point  $(x_i, y_j)$  the coefficients  $f_{k,l}$  can be written as

$$\begin{aligned} f(x_k, y_l) &= f_{i,j} + \frac{\partial f(x_i, y_j)}{\partial x} (x_k - x_i) + \frac{\partial f(x_i, y_j)}{\partial y} (y_l - y_j) \\ &\quad + \frac{1}{2} \frac{\partial^2 f(\xi_i^k, \zeta_j^l)}{\partial x^2} (x_k - x_i)^2 + \frac{\partial^2 f(\xi_i^k, \zeta_j^l)}{\partial xy} (x_k - x_i)(y_l - y_j) \\ &\quad + \frac{1}{2} \frac{\partial^2 f(\xi_i^k, \zeta_j^l)}{\partial y^2} (y_l - y_j)^2, \end{aligned}$$

the points  $(\xi_i^k, \zeta_j^l)$  being suitable points on the "extended" rectangular region  $R_{i,j}^{k,l} := [x_k, x_i] \times [y_l, y_j]$ . Now, because of the  $\mathbb{P}_0$  and the  $\mathbb{P}_1$  reproductions,  $\sigma$  reduces to

$$\begin{aligned} \sigma(x, y) &= f_{i,j} + \frac{\partial f(x_i, y_j)}{\partial x} (x_k - x_i) + \frac{\partial f(x_i, y_j)}{\partial y} (y_l - y_j) \\ &\quad + \frac{1}{2} \sum_{k=i-1}^{i+2} \sum_{l=j-1}^{j+2} \frac{\partial^2 f(\xi_i^k, \zeta_j^l)}{\partial x^2} (x_k - x_i)^2 C_k^x(x) C_l^y(y) \\ &\quad + \sum_{k=i-1}^{i+2} \sum_{l=j-1}^{j+2} + \frac{\partial^2 f(\xi_i^k, \zeta_j^l)}{\partial xy} (x_k - x_i)(y_l - y_j) C_k^x(x) C_l^y(y) \\ &\quad + \frac{1}{2} \sum_{k=i-1}^{i+2} \sum_{l=j-1}^{j+2} \frac{\partial^2 f(\xi_i^k, \zeta_j^l)}{\partial y^2} (y_l - y_j)^2 C_k^x(x) C_l^y(y). \end{aligned}$$

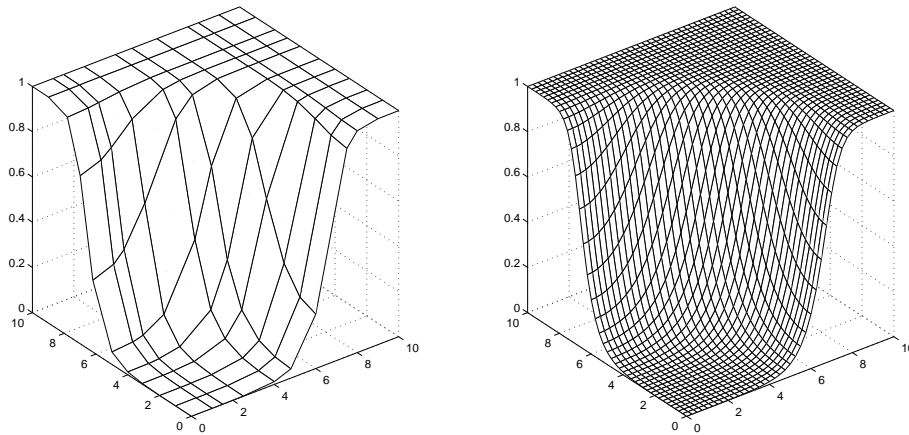
Thus, also expanding the function  $f$  around the point  $(x_i, y_j)$ , (12) holds true.

■

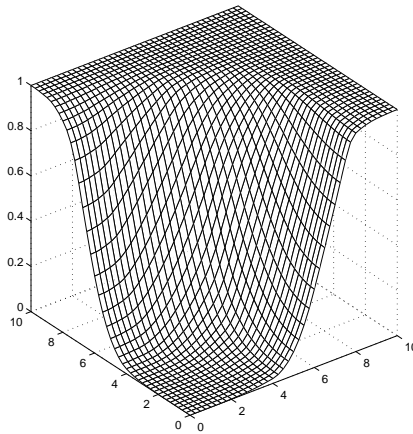
## §4. Examples

We conclude the paper with some examples showing how the proposed method could be used as a reliable tool for shape preserving quasi interpolation. In particular, we give two examples. In both cases the table of data is not explicitly reported but the data are depicted to put their shape in evidence.

The first example refers to non uniform gridded data from the sigmoidal function  $f(x, y) = (1 + 2e^{-3 \cdot (r-6.7)})^{-1/2}$  with  $r = (x^2 + y^2)^{1/2}$  taken from [1] and represented in figure 2 (left). The constructed quasi-interpolant function shown in figure 2 (right) emphasizes the goodness of the proposed method.



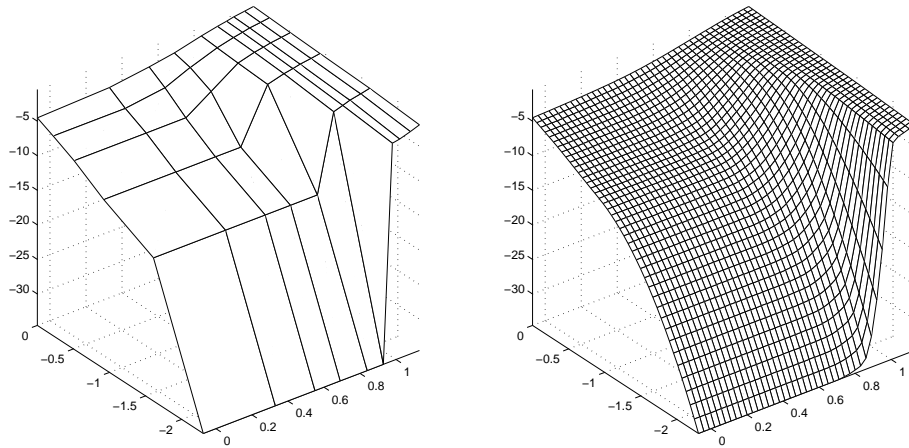
**Fig. 2.** *Example 1: the data and the sigmoidal function.*



**Fig. 3.** *Example 1: graph of the quasi-interpolant.*

The second example deals with the aluminum equation of state data [8]. The data and the quasi-interpolant function are shown in Figure 4 (left and right

respectively) where the reliability of the strategy is evident once again.



**Fig. 4.** *Example 2: the data and the quasi-interpolant.*

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# On the approximation error in Lagrange interpolation on equal intervals

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## Abstract

For a function  $f$  having a Maclaurin series expansion on  $I = [0, 1]$ , we present new formulas for the approximation error in Lagrange interpolation on equal intervals over  $I$ . These formulas allow to give simple sufficient conditions for the convergence and divergence of Lagrange interpolation.

**Keywords:** Lagrange interpolation, Approximation, Special functions, Combinatorics

## 1 Introduction

Lagrange interpolation on equal intervals is still under investigation: we cite here only two recent publications showing continuing interest in both its computational [3] and theoretical [8] aspects. One key issue is the convergence of the interpolating polynomial to the interpolated function. The aim of this paper is to investigate this subject and give sufficient conditions for which convergence or divergence hold.

Let  $f$  be a real function having a Maclaurin series expansion on  $I = [0, 1]$ , and let  $L_n$  be the Lagrange interpolating polynomial for  $f$  on the nodes  $\mathcal{N}_I = \{x_q = q/n, q = 0, \dots, n\}$ . We present here a new formula for the approximation error  $E_n(x) = f(x) - L_n(x)$ ,  $x \in I$ . Such formula does not involve, differently from the classical ones, the value of the  $(n+1)$ -th derivative of  $f$  at an unknown point. We also give an asymptotic expressions of the approximation error, that allows to obtain sharp bounds for  $E_n$ . One issue characterizing our approach is the use of some topics from number theory and combinatorics.

Formulas for the Lagrange interpolating polynomial on  $\mathcal{N}_I$  are well-known [11]; an alternative formula for  $L_n$  is [2]:

$$L_n(x) = \sum_{q=0}^n \sum_{k=0}^n (-1)^{k+q} \binom{k}{q} \binom{nx}{k} f(q/n), \quad x \in I. \quad (1)$$

Such expression follows from the approach in [3], but can be proved *a posteriori*, verifying that the interpolation conditions  $L_n(q/n) = f(q/n)$ ,  $q = 0, \dots, n$ , hold.

We recall here for future reference some well-known formulas. The fundamental formula for the approximation error, sometimes attributed to Cauchy, is:

$$E_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{q=0}^n (x - x_q), \quad x \in I,$$

for some unknown  $\xi \in (x_0, x_n)$ . For equally-spaced interpolation,  $x_q = q/n$ , it becomes:

$$E_n(x) = \frac{f^{(n+1)}(\xi)}{n^{n+1}} \binom{nx}{n+1}, \quad x \in I, \quad (2)$$

and, on defining  $M_{n+1} = \max_{x \in I} |f^{(n+1)}(x)|$ , an upper bound for  $E_n$  is [11]:

$$|E_n(x)| \leq \frac{M_{n+1}}{4(n+1)n^{n+1}}, \quad x \in I. \quad (3)$$

## 2 Main results

In this section we present the main results of our work; their proofs will be given in Section 3. The first result gives an alternative expression for the approximation error.

**Proposition 2.1** *Let  $f$  have a Maclaurin series expansion on  $I = [0, 1]$ . The approximation error in Lagrange interpolation on equal intervals can be written as:*

$$E_n(x) = \sum_{p=n+1}^{\infty} \sum_{k=n+1}^p \frac{f^{(p)}(0)}{p!} \frac{k!}{n^p} \left\{ \begin{matrix} p \\ k \end{matrix} \right\} \binom{nx}{k}, \quad x \in I, \quad (4)$$

where  $\left\{ \begin{matrix} p \\ k \end{matrix} \right\}$  are Stirling numbers of the second kind [6, 7].

Clearly, the above expression for  $E_n$  is not useful for practical purpose; however, it allows an asymptotic expression useful to analyze the convergence of Lagrange interpolation.

**Proposition 2.2** *Let  $E_n$  be given by Eq. (4). We have, for any  $x \in I$ :*

$$E_n(x) \sim \frac{f^{(n+1)}(1/2)}{n^{n+1}} \binom{nx}{n+1}, \quad n \rightarrow \infty. \quad (5)$$

On comparing Eq. (2) with Eq. (5), we see that Proposition 2.2 states that, for equally spaced points on  $I$ , and increasing  $n$ , the point  $\xi = \xi(x, x_0, \dots, x_n)$  in Eq. (2) tends to the center of the interpolation interval, independently on  $x$ . On defining:

$$E_n^\infty(x) = \frac{f^{(n+1)}(1/2)}{n^{n+1}} \binom{nx}{n+1}, \quad x \in I,$$

we see that this quantity depends on  $x$  only by the binomial; the next result gives the maximum of this function respect to  $x$ .

**Proposition 2.3** *For large  $n$ , the maximum of the function  $|E_n^\infty|$  is attained at a point  $x^*$  that tends to:*

$$x^* \sim \frac{1}{n(\ln(n) + \gamma)}, \quad n \rightarrow \infty,$$

where  $\gamma = 0.577215665$  is Euler's constant [7].

Combining Propositions 2.2 and 2.3 we obtain the expression, for large  $n$ , of the maximum of the approximation error over the whole interval  $I$ :

$$\overline{E}_n^\infty = \max_{x \in I} |E_n^\infty(x)| = \left| \frac{f^{(n+1)}(1/2)}{n^{n+1}} \left( \frac{1}{\ln(n) + \gamma} \right) \right|. \quad (6)$$

To give a characterization of the class of functions for which convergence of Lagrange interpolation on equal intervals holds, we give a bound on the binomial in Eq. (6); we have:

**Proposition 2.4** *The binomial in Eq. (6) tends to zero as  $n$  increases, bounded by:*

$$\frac{1}{n^2 \ln(n)} < \left| \left( \frac{1}{\ln(n) + \gamma} \right) \right| < \frac{1}{n \ln(n)}. \quad (7)$$

We are now able to prove our main result about the convergence of Lagrange interpolation. By Eq. (6) and Proposition 2.4 the following holds.

**Proposition 2.5** *Let  $f$  have a Maclaurin series expansion on  $I$ . Sufficient conditions for convergence and divergence of Lagrange interpolation on equal intervals are, respectively:*

$$f^{(n+1)}(1/2) \prec n^{n+2} \ln(n), \quad (8)$$

$$f^{(n+1)}(1/2) \succ n^{n+3} \ln(n), \quad (9)$$

where the notation  $a(n) \prec b(n)$  means  $\lim_{n \rightarrow \infty} a(n)/b(n) = 0$  [7, pp. 426].

It is well-known that the maximum of  $|E_n(x)|$  may diverge for increasing  $n$  when the function is interpolated on equal intervals on  $I$ . This is known as *Runge's phenomenon*, and it is explained by introducing an integral representation of  $|E_n|$ , and showing that this diverges for increasing  $n$  as the extension of  $f$  to the complex field has poles too close to  $I$  [5, 9]. Proposition 2.5 may be viewed as an alternative explanation of Runge's phenomenon.

### 3 Proofs of Proposition 2.1 through 2.4

#### 3.1 PROOF OF PROPOSITION 2.1

Taking the Maclaurin series of  $f$  and substituting in (1) we have:

$$L_n(x) = \sum_{p=0}^{\infty} \sum_{q=0}^n \sum_{k=0}^n (-1)^{k+q} \frac{f^{(p)}(0)}{p!} \binom{k}{q} \binom{nx}{k} (q/n)^p;$$

now we use two properties of Stirling numbers [7]:

$$\begin{aligned} \sum_{q=0}^k (-1)^q q^p \binom{k}{q} &= (-1)^k k! \left\{ \begin{matrix} p \\ k \end{matrix} \right\}, \\ \sum_{k=0}^p k! \left\{ \begin{matrix} p \\ k \end{matrix} \right\} \binom{nx}{k} &= (nx)^p, \end{aligned}$$

and, taking into account that  $\left\{ \begin{matrix} p \\ k \end{matrix} \right\} = 0$  for  $k > p$ , we get:

$$L_n(x) = \sum_{p=0}^{\infty} \frac{f^{(p)}(0)}{p!} x^p - \sum_{p=n+1}^{\infty} \sum_{k=n+1}^p \frac{f^{(p)}(0)}{p!} \frac{k!}{n^p} \left\{ \begin{matrix} p \\ k \end{matrix} \right\} \binom{nx}{k}. \quad \square$$

### 3.2 PROOF OF PROPOSITION 2.2

Let us rewrite Eq. (4) as

$$E_n(x) = \sum_{p=0}^{\infty} \sum_{k=0}^p \frac{f^{(p+n+1)}(0)}{n^{p+n+1}} \frac{(p+n+1-k)!}{(p+n+1)!} \left\{ \begin{matrix} p+n+1 \\ p+n+1-k \end{matrix} \right\} \binom{nx}{p+n+1-k};$$

using an asymptotic expansion of Stirling numbers of the second kind [6]:

$$\left\{ \begin{matrix} p+n+1 \\ p+n+1-k \end{matrix} \right\} \sim \frac{(p+n+1)^{2k}}{2^k k!}, \quad n \rightarrow \infty,$$

we get, for large  $n$ :

$$E_n(x) \sim \frac{1}{n^{n+1}} \sum_{p=0}^{\infty} \sum_{k=0}^p \frac{(p+n+1-k)!}{(p+n+1)!} \frac{(p+n+1)^{2k}}{n^p} \frac{f^{(p+n+1)}(0)}{2^k k!} \binom{nx}{p+n+1-k}.$$

Now note that, for fixed  $k$  and  $p$ ,

$$\frac{(p+n+1-k)!}{(p+n+1)!} \frac{(p+n+1)^{2k}}{n^p} \sim \begin{cases} 0, & \text{if } k < p, \\ 1, & \text{if } k = p, \end{cases} \quad n \rightarrow \infty,$$

that implies:

$$E_n(x) \sim \frac{1}{n^{n+1}} \binom{nx}{n+1} \sum_{p=0}^{\infty} \frac{f^{(p+n+1)}(0)}{2^p p!}, \quad n \rightarrow \infty.$$

The sum of the series is nothing but  $f^{(n+1)}(1/2)$ , and this concludes the proof.  $\square$

### 3.3 PROOF OF PROPOSITION 2.3

Since the binomial attains its extrema in the first and last sub-intervals, and its absolute value is symmetric about the middle point of  $I$  [11], one has:

$$\arg \max_{x \in I} |E_n^{\infty}(x)| = \arg \max_{x \in (0, \frac{1}{n})} \left| \binom{nx}{n+1} \right|;$$

moreover, from [1]:

$$\binom{nx}{n+1} = \frac{\Gamma(nx+1)}{\Gamma(n+2) \Gamma(nx-n)},$$



one can write

$$\frac{d}{dx} \binom{nx}{n+1} = n \binom{nx}{n+1} (\psi(nx+1) - \psi(nx-n)),$$

where the function  $\psi$  is the logarithmic derivative of Euler's Gamma function  $\Gamma$ . Thus, the above derivative vanishes at a point  $x^* \in (0, 1/n)$  where:

$$\psi(nx^* + 1) = \psi(nx^* - n).$$

Owing to the following properties of the function  $\psi$  [4]:

$$\begin{aligned} \psi(nx+1) &= \psi(nx) + \frac{1}{nx}, \\ \psi(nx-n) &= \psi(nx) - \sum_{k=1}^n \frac{1}{nx-k}, \end{aligned}$$

the point  $x^*$  we are seeking for satisfies:

$$\frac{1}{nx^*} + \sum_{k=1}^n \frac{1}{nx^* - k} = 0.$$

On introducing the Maclaurin series,

$$\frac{1}{nx-k} = \sum_{p=1}^{\infty} \frac{1}{k^p} (nx)^{p-1},$$

and the harmonic number of order  $p$  [7]:

$$H_n^{(p)} = \sum_{k=1}^n \frac{1}{k^p}, \quad p \in \mathbf{N},$$

one finally gets:

$$H_n^{(1)} \left( 1 + \sum_{p=2}^{\infty} \frac{H_n^{(p)}}{H_n^{(1)}} (nx^*)^{p-1} \right) = \frac{1}{nx^*}.$$

Note that, when  $n$  grows,  $H_n^{(1)} \sim \ln(n) + \gamma$ , and  $H_n^{(p)} \sim \zeta(p) \leq \zeta(2) = \frac{\pi^2}{6}$ ,  $p \geq 2$ , where  $\zeta$  is Riemann's Zeta function [7]; thus we have:

$$0 < \sum_{p=2}^{\infty} \frac{H_n^{(p)}}{H_n^{(1)}} (nx^*)^{p-1} < \frac{\zeta(2)}{\ln(n) + \gamma} \frac{nx^*}{1 - nx^*},$$

therefore, the equation to solve becomes, for large  $n$ ,  $H_n^{(1)} = 1/nx^*$ , whose solution is  $x^* = \left( n(\ln(n) + \gamma) \right)^{-1}$ .  $\square$

### 3.4 PROOF OF PROPOSITION 2.4

Let us define, to simplify the notation,  $\theta_n = \ln(n) + \gamma$ ; note that  $\theta_n > 1$  for  $n > 1$ . We have:

$$\begin{aligned} \left| \binom{1/\theta_n}{n+1} \right| &= \prod_{k=1}^{n+1} \frac{|1/\theta_n - k + 1|}{k} \\ &= \frac{1}{\theta_n} \prod_{k=2}^{n+1} \frac{(k-1)\theta_n - 1}{k\theta_n} \\ &< \frac{1}{\theta_n} \prod_{k=2}^{n+1} \frac{k-1}{k} \\ &= \frac{1}{\theta_n} \frac{1}{n+1}. \end{aligned}$$

Furthermore, we have:

$$\begin{aligned} \left| \binom{1/\theta_n}{n+1} \right| &= \frac{\theta_n - 1}{2\theta_n^2} \prod_{k=3}^{n+1} \frac{(k-1)\theta_n - 1}{k\theta_n} \\ &> \frac{\theta_n - 1}{2\theta_n^2} \prod_{k=3}^{n+1} \frac{(k-1)\theta_n - \theta_n}{k\theta_n} \\ &= \frac{\theta_n - 1}{2\theta_n^2} \prod_{k=3}^{n+1} \frac{k-2}{k} \\ &= \frac{\theta_n - 1}{n(n+1)\theta_n^2}, \end{aligned}$$

thus, for large  $n$ :

$$\frac{\ln(n) + \gamma - 1}{n(n+1)(\ln(n) + \gamma)^2} < \left| \binom{1/\theta_n}{n+1} \right| < \frac{1}{(n+1)(\ln(n) + \gamma)} < \frac{1}{n \ln(n)}.$$

Now, for large  $n$ ,

$$\frac{\ln(n) + \gamma - 1}{\ln(n) + \gamma} \sim 1,$$

and

$$\frac{1}{n(n+1)} \sim \frac{1}{n^2},$$

thus Eq. (7) holds for sufficiently large  $n$ . Though the previous equivalences are asymptotic, the numerical experience shows that (7) holds for any  $n > 1$ .  $\square$

## 4 Some examples

For comparison, we compute here the proposed bounds (6) for two simple functions and the corresponding bounds obtained by classical formula (3).

**Corollary 4.1** *The maximum approximation error for  $f(x) = \exp(ax)$  is given by:*

$$\rho_n = \max_{x \in I} |\exp(ax) - L_n(x)| \sim \left| \left( \frac{1}{\ln(n) + \gamma} \right) \frac{a^{n+1}}{n^{n+1}} \right| \exp(a/2), \quad n \rightarrow \infty. \quad (10)$$

**Corollary 4.2** *The maximum approximation error for  $f(x) = \sin(ax)$  is given by:*

$$\sigma_n = \max_{x \in I} |\sin(ax) - L_n(x)| \sim \left| \left( \frac{1}{\ln(n) + \gamma} \right) \frac{a^{n+1}}{n^{n+1}} \sin(a/2 + \mu_n) \right|, \quad n \rightarrow \infty, \quad (11)$$

where  $\mu_n = 0$  for even  $n$  and  $\mu_n = -\pi/2$  for odd  $n$ .

When we specialize Eq. (3) to the exponential function we get, for positive  $a$ :

$$\alpha_n = \left| \frac{a^{n+1}}{n^{n+1}} \frac{\exp(a)}{4(n+1)} \right|;$$

for negative  $a$  simply replace  $\exp(a)$  by one. At the same way, for  $f(x) = \sin(ax)$  we get:

$$\beta_n = \left| \frac{a^{n+1}}{n^{n+1}} \frac{1}{4(n+1)} \right|.$$

By Eq. (7) we see that the estimates of the approximation error given by (10) and (11) are, for sufficiently large  $n$ , tighter than the bounds one can get from (3). In particular:

$$\rho_n/\alpha_n < \frac{4 \exp(-a/2)}{\ln(n) + \gamma}, \quad \sigma_n/\beta_n < \frac{4 \sin(a/2 + \mu_n)}{\ln(n) + \gamma}.$$

True ratios  $\rho_n/\alpha_n$  and  $\sigma_n/\beta_n$  are actually smaller than those predict by the above formulas, as bound (7) is not tight. We have numerically verified that the binomial in (7) is smaller than  $1/(4(n+1))$  for  $n \leq 30$ , and this bound is tighter than that given by (7) in such range. As an example, with  $n = 15$  and  $a = 4$ , we have:  $\alpha_n = 5.578 \cdot 10^{-10}$ ;  $\beta_n = 1.022 \cdot 10^{-11}$ ;  $\rho_n = 3.067 \cdot 10^{-11}$ ;  $\sigma_n = 1.727 \cdot 10^{-12}$ , and it results  $\rho_n/\alpha_n = 0.055$  and  $\sigma_n/\beta_n = 0.169$ .

Let us now define the following three indexes:

$$\text{TEI}_n \quad (\text{True Error Index}): \quad \text{TEI}_n = \|E_n\|_1;$$

$$\text{EEI}_n \quad (\text{Estimated Error Index}): \quad \text{EEI}_n = \|E_n^\infty\|_1;$$

$$\text{PEI}_n \quad (\text{Percentage Error Index}) \quad \text{PEI}_n = 100 (\text{TEI}_n - \text{EEI}_n)/\text{TEI}_n.$$

In Figure 1 we plot some values of  $\text{TEI}_n$ ,  $\text{EEI}_n$  and  $\text{PEI}_n$  for the exponential function with  $a = 1/2$ . We can see that the estimated index ( $\text{EEI}_n$ ) shows a good agreement with the true index ( $\text{TEI}_n$ ), and converges monotonically to this last as  $n$  increases. The computation has been performed in arbitrary precision using the package **Mathematica** [10].

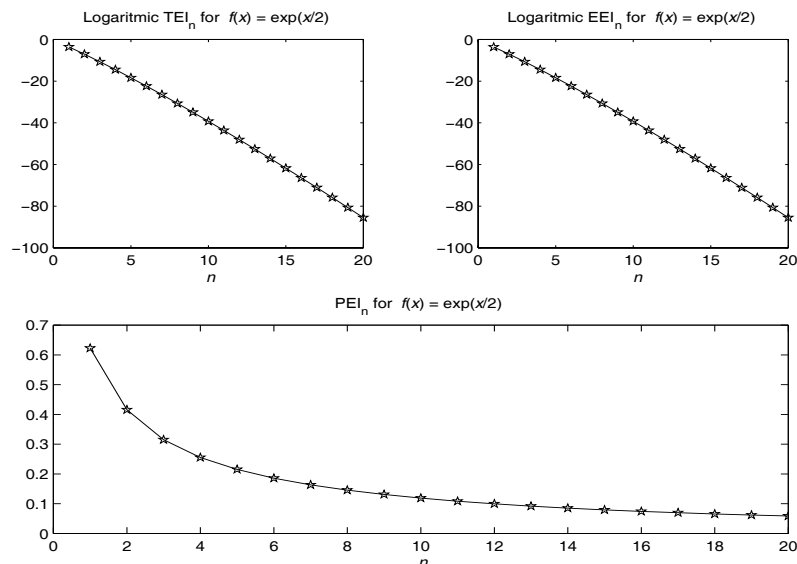


Figure 1: Plots of the indexes  $TEI_n$ ,  $EEI_n$  and  $PEI_n$  for  $n = 1, \dots, 20$

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## A Strong Balian-Low Theorem<sup>1</sup>

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### Abstract

The Balian-Low theorem is a key result in time-frequency analysis. It says that if a Gabor system  $\{e^{2\pi i m t} g(t - n)\}$  forms a frame for  $L^2(R)$ , then either  $tg(t)$  or  $\omega \hat{g}(\omega)$  can not be in  $L^2(R)$ . In this paper, a new theorem will be proved for a family of non-uniform Gabor frames  $\{e^{2\pi i x_m t} g(t - y_n)\}$  with Condition  $G$ , which is stronger than Balian-Low theorem.

**Subject Classification:** 42C15

**Key words:** Gabor frame, Gabor basis, density, wavelet.

## 1 Introduction and Preliminary

Frames play an important role in both applied mathematics and engineering. A frame of  $L^2(R)$  means a sequence  $\{g_n\} \subseteq L^2(R)$  such that for each  $f \in L^2(R)$ , one has

$$A\|f\|^2 \leq \sum_n |\langle f, g_n \rangle|^2 \leq B\|f\|^2,$$

where  $A$  and  $B$  called lower and upper frame bounds respectively. If  $A = B$ , the frame is said to be tight. It is well known that the frame concept is a generalization of a Riesz basis. In fact  $\{g_n\}$  is a Riesz basis if and only if  $\{g_n\}$  is both a frame and  $l^2$  linear independent.

For  $g \in L^2(R)$  and two real sequences  $\{x_m\}, \{y_n\}$ , if  $\{e^{2\pi i x_m t} g(t - y_n)\}$  forms a frame for  $L^2(R)$ , then it is called a Gabor frame or sometimes Weyl-Heisenberg frame. Similarly we call  $\{e^{2\pi i x_m t} g(t - y_n)\}$  a Gabor basis, when the system constitutes a Riesz basis  $L^2(R)$ . When  $x_m = 2\pi m b$  and  $y_n = n a$ , where  $m, n$  are integers and  $a, b$  are positive constants, the corresponding Gabor frame is said to be uniform, otherwise non-uniform. A typical example for uniform Gabor frames is  $\{e^{2\pi i m t} \chi_{[0,1)}(t - n)\}$ , where  $a = b = 1$  and  $g(t) = \chi_{[0,1)}(t)$  is the characteristic function on the interval  $[0, 1)$ . In fact, it is an orthonormal basis and therefore a tight frame for  $L^2(R)$ . Daubechies, Grossman and Meyer ([5]) gave more general construction

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for  $\{e^{2\pi imbt}g(t-na)\}$  to be a tight frame of  $L^2(R)$  in 1986, where

$$g(t) = a^{1/2} \begin{cases} \sin[\frac{\pi}{2}v(\frac{1+2t}{1-ab})], & -\frac{1}{2b} \leq t < \frac{1}{2b} - a \\ 1, & \frac{1}{2b} - a \leq t < -\frac{1}{2b} + a \\ \cos[\frac{\pi}{2}v(\frac{1-2t}{1-ab})], & -\frac{1}{2b} + a \leq t < \frac{1}{2b} \\ 0, & t \notin [-\frac{1}{2b}, \frac{1}{2b}] \end{cases}$$

for certain functions  $v$  and positive numbers  $a, b$  with  $1/2 < ab < 1$ . Note that the support of  $g$ ,  $\text{supp } g =: \{t \in R, g(t) \neq 0\} \subseteq [-1/2b, 1/2b]$  and  $\{\sqrt{b}e^{2\pi imbt}\}$  is an orthonormal basis of  $L^2[-1/2b, 1/2b]$ . More generally Benedetto and Walnut ([2]) proved the followings in 1994:

**Theorem 1.1** *Suppose that  $g \in L^2(R)$  is a bounded function with  $\text{supp } g \subseteq [0, 1/b]$ . Then  $\{e^{2\pi imbt}g(t-na)\}$  is a Gabor frame if and only if*

$$0 < A \leq \sum_n |g(t-na)|^2 \leq B < +\infty$$

*holds almost everywhere.*

Again  $g$  is compactly supported in  $[0, 1/b]$  and  $\{\sqrt{b}e^{2\pi imbt}\}$  is an orthonormal basis of  $L^2[0, 1/b]$  in this theorem. There are many research achievements about non-uniform Gabor frames (see [4], [7], [9] and [10]), we mention the following result taken from [8], which will be used in this paper.

**Theorem 1.2** *Let  $\{e^{2\pi ix_mt}\}$  be a frame of  $L^2[\beta, \gamma]$  and  $g \in L^2(R)$  be a bounded function with  $\text{supp } g \subseteq [\beta, \gamma]$ . Then  $\{e^{2\pi ix_mt}g(t-y_n)\}$  is a Gabor frame if and only if*

$$c \leq \sum_n |g(t-y_n)|^2 \leq C$$

*holds almost everywhere for some  $C > c > 0$ .*

It is clear that Theorem 1.2 is a complete generalization of Theorem 1.1 from uniform case to non-uniform case. Now we introduce a new concept— **Condition  $G$**  for convenience.

**Definition 1.1** A non-uniform Gabor system  $\{e^{2\pi ix_mt}g(t-y_n)\}$  is said to satisfy Condition  $G$ , if  $g \in L^2(R)$  is bounded and compactly supported on  $[\beta, \gamma]$ ,  $\{e^{2\pi ix_mt}\}$  is a frame of  $L^2[\beta, \gamma]$ . Similarly it is called a Gabor frame (basis) with Condition  $G$ , if the Gabor frame (basis) satisfies those conditions.

By this terminology, Theorem 1.2 is given in another form:

**Theorem 1.2'** A Gabor system  $\{e^{2\pi ix_mt}g(t-y_n)\}$  with Condition  $G$  is a Gabor frame if and only if

$$0 < c \leq \sum_n |g(t-y_n)|^2 \leq C$$

holds almost everywhere.

In this paper, we shall show a strong Balian-Low theorem for Gabor frames with Condition  $G$ . The well known Balian-Low theorem says that if  $\{e^{2\pi i m b t} g(t - na)\}$  with  $ab = 1$  constitutes a Gabor frame, then either  $\int t^2 |g(t)|^2 dt = +\infty$  or  $\int \omega^2 |\hat{g}(\omega)|^2 d\omega = +\infty$ , where  $\hat{g}(\omega)$  is the Fourier transform of  $g(t)$ . Since  $\{e^{2\pi i m b t} g(t - na)\}$  with  $ab = 1$  being a Gabor frame is equivalent to that the system is a Gabor basis ([1], [3]), the Balian-Low theorem can be restated as follows:

**Theorem 1.3** *Let  $\{e^{2\pi i m b t} g(t - na)\}$  be a Gabor basis. Then  $\|tg(t)\| \cdot \|\omega \hat{g}(\omega)\| = +\infty$ . Here  $\|f\|^2 = \int |f(t)|^2 dt$ .*

We don't know if the same result is true for any non-uniform Gabor bases. But it is for each Gabor basis with Condition  $G$ . In fact we can say a little more, which is the main theorem in this paper.

**Theorem 1.4** *Let  $\{e^{2\pi i x_m t} g(t - y_n)\}$  be a Gabor basis with Condition  $G$ . Then  $g(t)$  can not be continuous.*

It can be seen that this above theorem is stronger than what Balian-Low theorem says for those Gabor bases with Condition  $G$  (see Corollary 3.2). To prove this result, we need several notations.

Let  $B$  denote the unit square centered at the origin in  $R^d$ , i. e.  $B = \{x = (x_1, x_2, \dots, x_d), |x_i| \leq 1/2, 1 \leq i \leq d\}$ . Given  $r > 0$ ,  $x \in R^d$  and a discrete subset  $A$  of  $R^d$ , define  $S(x, r) =: \{x + rt, t \in B\}$  and  $n(S(x, r), A) =:$  the number of points in  $A \cap S(x, r)$ . Furthermore define  $n^+(r, A) =: \sup_{x \in R^d} n(S(x, r), A)$  and  $n^-(r, A) =: \inf_{x \in R^d} n(S(x, r), A)$ .

**Definition 1.2** The uniform upper and lower density of a set  $A$  in  $R^d$  are defined by

$$D^+(A) = \overline{\lim}_{r \rightarrow \infty} \frac{n^+(r, A)}{r^d}$$

and

$$D^-(A) = \underline{\lim}_{r \rightarrow \infty} \frac{n^-(r, A)}{r^d}$$

respectively. If these two quantities are the same, then the set  $A$  is said to have uniform density given by

$$D(A) = D^-(A) = D^+(A).$$

H. J. Landau has shown these quantities are unchanged if the the unit square  $B$  is replaced by any other bounded set with measure 1. In many references, the set is taken as the unit ball of  $R^d$  in the sense of Euclidean norm.

**Definition 1.3** Let  $X = \{x_n\}$  be a real sequence. If  $|x_n - x_m| \geq \delta > 0$  for each  $n \neq m$ , then  $X$  is said to be separated;  $X$  is called relatively separated, if it is a finite union of separated sequences. It is easy to show (see [4]) the followings:

**Theorem 1.5** *Let  $X = \{x_n\}$  be a real sequence. Then the following statements are equivalent:*

1.  $D^+(X) < +\infty$ .
2.  $X$  is relatively separated.
3. For each  $s > 0$ , there exists a positive integer  $N = N(s)$  such that at most  $N$  elements of  $X$  are inside each interval of length  $s$ .

In Section 2, we shall show that if  $\{e^{2\pi i x_m t} g(t - y_n)\}$  is a Gabor basis with Condition  $G$ , then  $D(X) = \gamma - \beta$  and  $D(Y) = 1/(\gamma - \beta)$  with  $X = \{x_m\}$  and  $Y = \{y_n\}$ . By this result the main theorem will be proved in the last section.

## 2 A Density Property of Gabor Bases

In this section, a density property of Gabor bases with Condition  $G$  will be given. We always assume that  $X = \{x_m\}, Y = \{y_n\} \subseteq \mathbb{R}$  are real sequences and  $A = X \times Y \subseteq \mathbb{R}^2$  is the usual Cartesian product in this work. Then we have

**Lemma 2.1** *If both  $D^+(X) < +\infty$  and  $D^+(Y) < +\infty$ , then*

$$D^+(A) = D^+(X)D^+(Y)$$

and

$$D^-(A) = D^-(X)D^-(Y).$$

Proof. One only gives a detailed proof of  $D^+(A) = D^+(X)D^+(Y)$ . Since  $D^+(X) < +\infty$ , one knows that  $n(S(x_1, r), X)$  is finite for any real numbers  $x_1$  and  $r > 0$  by Theorem 1.5. Similarly  $n(S(x_2, r), Y) < +\infty$ . Furthermore

$$n(S(x, r), A) = n(S(x_1, r), X).n(S(x_2, r), Y)$$

and

$$n^+(r, A) \leq n^+(r, X).n^+(r, Y)$$

for each  $x = (x_1, x_2) \in \mathbb{R}^2$  and  $A = X \times Y$ . It then follows  $D^+(A) \leq D^+(X)D^+(Y)$  from Definition 1.2. Hence one only need show

$$D^+(A) \geq D^+(X)D^+(Y).$$

Due to Definition 1.2, there exist two sequences of intervals  $I_{r_n}$  and  $J_{s_n}$  with length  $r_n$  and  $s_n$  respectively ( $I_a$  and  $J_a$  always denote some intervals with length  $a$  here and afterwards) such that

$$\lim_{n \rightarrow \infty} \frac{n(I_{r_n}, X)}{r_n} = D^+(X) \quad (2.1)$$



and

$$\lim_{n \rightarrow \infty} \frac{n(J_{s_n}, Y)}{s_n} = D^+(Y), \quad (2.2)$$

where  $1 \leq r_n \leq r_{n+1} \rightarrow +\infty$  and  $1 \leq s_n \leq s_{n+1} \rightarrow +\infty$ . Without loss of generality one assumes  $r_n \geq s_n^2$ , since  $r_n$  can be replaced by its subsequence, if needed.

For fixed interval  $I_r$  with length  $r$  and  $0 < s \leq r$ , one can pick up some subinterval  $I_s$  of  $I_r$  such that  $n(I_s, X) \geq n(J_s, X)$  holds for each subinterval  $J_s$  of  $I_r$ , because of  $D^+(X) < +\infty$  and Theorem 1.5. Then it follows that

$$n(I_r, X) \leq (r/s + 1)n(I_s, X).$$

Now for  $0 < s_n \leq r_n$  and  $I_{r_n}$ , one has  $I_{s_n} \subseteq I_{r_n}$  and

$$n(I_{r_n}, X) \leq (r_n/s_n + 1)n(I_{s_n}, X),$$

which is equivalent to

$$\frac{n(I_{r_n}, X)}{r_n} - \frac{n(I_{s_n}, X)}{r_n} \leq \frac{n(I_{s_n}, X)}{s_n}. \quad (2.3)$$

Note that  $\lim_n \frac{s_n}{r_n} = 0$ , since  $r_n \geq s_n^2$  and  $\lim_n s_n = \infty$ . Furthermore from the assumed condition  $D^+(X) < +\infty$  and

$$0 \leq \frac{n(I_{s_n}, X)}{r_n} \leq \frac{n(I_{s_n}, X)}{s_n} \frac{s_n}{r_n},$$

one can conclude

$$\lim_n \frac{n(I_{s_n}, X)}{r_n} = 0. \quad (2.4)$$

Now it follows

$$\varliminf_n \frac{n(I_{s_n}, X)}{s_n} \geq \lim_n \left[ \frac{n(I_{r_n}, X)}{r_n} - \frac{n(I_{s_n}, X)}{r_n} \right] = D^+(X)$$

from (2.3), (2.4) and (2.1). Furthermore one has

$$\lim_n \frac{n(I_{s_n}, X)}{s_n} = D^+(X) \quad (2.5)$$

by the definition of  $D^+(X)$ . Finally one obtains

$$D^+(A) \geq \varliminf_n \frac{n(I_{s_n} \times J_{s_n}, A)}{s_n^2} = \lim_n \frac{n(I_{s_n}, X)}{s_n} \cdot \lim_n \frac{n(J_{s_n}, Y)}{s_n} = D^+(X)D^+(Y)$$

because of (2.2) and (2.5). This completes the proof of  $D^+(A) = D^+(X)D^+(Y)$ .

The proof of  $D^-(A) = D^-(X)D^-(Y)$  is similar. In fact one only need show

$$D^-(A) \leq D^-(X)D^-(Y).$$

It is reasonable to assume

$$\lim_{n \rightarrow \infty} \frac{n(I_{r_n}, X)}{r_n} = D^-(X)$$

and

$$\lim_{n \rightarrow \infty} \frac{n(J_{s_n}, Y)}{s_n} = D^-(Y)$$

with  $r_n \geq s_n^2$  as in the first part. Again take some subinterval  $I_{s_n}$  of  $I_{r_n}$  with

$$(r_n/s_n - 1)n(I_{s_n}, X) \leq n(I_{r_n}, X).$$

Here one should choose subinterval  $I_s$  of  $I_r$  with the property that  $n(I_s, X) \leq n(J_s, X)$  for each subinterval  $J_s$  of  $I_r$ . By showing

$$\overline{\lim}_n \frac{n(I_{s_n}, X)}{s_n} \leq D^-(X),$$

one can conclude  $\lim_n \frac{n(I_{s_n}, X)}{s_n} = D^-(X)$ . Finally it follows

$$D^-(A) \leq D^-(X)D^-(Y),$$

as desired.

To introduce Lemma 2.3, we need the following result from [4].

**Lemma 2.2** *If  $\{e^{2\pi i x_m t} g(t - y_n)\}$  is a Gabor frame, then  $1 \leq D^-(A) \leq D^+(A) < +\infty$ ; Furthermore if  $\{e^{2\pi i x_m t} g(t - y_n)\}$  is a Gabor basis, then  $D(A) = 1$ . Here  $A = \{x_m\} \times \{y_n\}$ .*

**Lemma 2.3** *Let  $\{e^{2\pi i x_m t} g(t - y_n)\}$  be a Gabor frame. Then  $0 < D^-(X) \leq D^+(X) < +\infty$  and  $0 < D^-(Y) \leq D^+(Y) < +\infty$  with  $X = \{x_m\}$  and  $Y = \{y_n\}$ . Furthermore under some rearrangement of  $X$  and  $Y$ , one has*

$$-\infty \longleftarrow x_m < x_{m+1} \longrightarrow +\infty$$

and

$$-\infty \longleftarrow y_n < y_{n+1} \longrightarrow +\infty.$$

Proof: To show  $D^-(X) > 0$ , it is sufficient to prove that there exists some  $L > 0$  such that  $X \cap I_L \neq \emptyset$  for each interval  $I_L$  with length  $L$ . Suppose not, then there would have some interval sequence  $\{I_n\}$  of length  $n$  with the property  $X \cap I_n = \emptyset$ . This implies  $(I_n \times I_n) \cap A = \emptyset$  and therefore

$$0 \leq D^-(A) \leq \underline{\lim}_n \frac{n(I_n \times I_n, A)}{n^2} = 0$$

with  $A = X \times Y$ . On the other hand, since  $\{e^{2\pi i x_m t} g(t - y_n)\}$  is a Gabor frame, one has  $D^-(A) \geq 1$  by Lemma 2.2. This is a contradiction. Similarly  $D^-(Y) > 0$  is proved.

Next one shows  $D^+(X) < +\infty$  by contradiction again: Suppose  $D^+(X) = +\infty$ , then there would have some sequence of intervals  $\{I_{r_n}\}$  with length  $r_n$  such that  $n(I_{r_n}, X)/r_n > n$ . Then it follows

$$D^+(A) \geq \overline{\lim}_n \frac{n(I_{r_n} \times I_{r_n}, A)}{r_n^2} = \overline{\lim}_n \left[ \frac{n(I_{r_n}, X)}{r_n} \frac{n(I_{r_n}, Y)}{r_n} \right] = +\infty$$

from the proved fact  $D^-(Y) > 0$  and  $\frac{n(I_{r_n, X})}{r_n} > n$ , which contradicts with Lemma 2.2. The similar arguments lead to  $D^+(Y) < +\infty$ .

Finally since  $D^+(X) < +\infty$ , there exists some positive integer  $M$  such that the number of points in  $X \cap [n, n+1]$  is not larger than  $M$  for each integer  $n \in \mathbb{Z}$  by Theorem 1.5. Now one can rearrange  $\{x_m\}$  (if necessary) such that  $x_m < x_{m+1}$  for each  $m$ . Furthermore  $D^-(X) > 0$  implies  $\lim_{m \rightarrow +\infty} x_m = +\infty$  and  $\lim_{m \rightarrow -\infty} x_m = -\infty$ . Altogether one has

$$-\infty \longleftarrow x_m < x_{m+1} \longrightarrow +\infty.$$

Similarly

$$-\infty \longleftarrow y_n < y_{n+1} \longrightarrow +\infty.$$

Due to this lemma, we can always assume  $x_m < x_{m+1}$  and  $y_n < y_{n+1}$ , when  $\{e^{2\pi i x_m t} g(t - y_n)\}$  is a Gabor frame. Now we are ready to state the following theorem:

**Theorem 2.1** *Let  $\{e^{2\pi i x_m t} g(t - y_n)\}$  be a Gabor basis. Then  $D^+(X) = D^-(X)$ ,  $D^+(Y) = D^-(Y)$  and  $D(X)D(Y) = 1$ , where  $X = \{x_m\}$  and  $Y = \{y_n\}$ .*

Proof: According to Lemma 2.3, one has  $D^+(X) < +\infty$  and  $D^+(Y) < +\infty$ . Now since all conditions of Lemma 2.1 are satisfied and  $D(A) = 1$  follows from Lemma 2.2, one can conclude

$$D^+(X)D^+(Y) = D^+(A) = 1$$

and

$$D^-(X)D^-(Y) = D^-(A) = 1.$$

Note that  $D^-(X) \leq D^+(X)$  and  $D^-(Y) \leq D^+(Y)$ , the desired conclusion follows.

The following lemma gives a characterization for a family of Gabor frames to be complete in  $L^2(R)$ . We include the proof for completeness and interests, although only one direction of the result is used in this paper.

**Lemma 2.4** *Let  $S$  be a measurable set of  $R$  and  $g \in L^2(R)$  be a bounded function with  $\text{supp } g = S$ . If  $\{e^{2\pi i x_m t}\}$  is complete in  $L^2(S)$ , then  $\{g_{mn}(t) =: e^{2\pi i x_m t} g(t - y_n)\}$  is complete in  $L^2(R)$  if and only if*

$$\cup_n (S + y_n) = R$$

*holds almost everywhere.*

Proof: For the necessary part, one use contradiction. Suppose not, then there would exist some measurable set  $E_0 \subseteq R \setminus \cup_n (S + y_n)$  with  $0 < \mu(E_0) < +\infty$ . It is clear that  $E_0 \cap (S + y_n) = \emptyset$  or  $(E_0 - y_n) \cap S = \emptyset$  for each  $n \in \mathbb{Z}$ .

Define  $f_0(t) =: \chi_{E_0}(t)$ , the characteristic function of  $E_0$ . Then  $0 \neq f_0 \in L^2(R)$  due to  $0 < \mu(E_0) < +\infty$ . Note that  $\text{supp } g = S$  and  $(E_0 - y_n) \cap S = \emptyset$ , one has

$$\langle f_0, g_{mn} \rangle = \int_{E_0} e^{-2\pi i x_m t} \overline{g(t - y_n)} dt = 0.$$

This contradicts with the completeness of  $\{g_{mn}\}$ .

For the sufficient part, one shows a simple claim first:  $\{e^{2\pi i x_m t}\}$  is complete in  $L^2(S + a)$  for each real number  $a \in R$ . In fact for any  $f \in L^2(S + a)$ , one knows  $f(t + a) \in L^2(S)$ . Suppose

$$\langle f(t), e^{2\pi i x_m t} \rangle_{S+a} = 0,$$

where  $\langle f, g \rangle_S$  denotes the inner product in the space  $L^2(S)$ . Then

$$0 = \int_{S+a} f(t) e^{-2\pi i x_m t} dt = \int_S f(\mu + a) e^{-2\pi i x_m (\mu + a)} d\mu = e^{-2\pi i x_m a} \langle f(t + a), e^{2\pi i x_m t} \rangle.$$

Since  $\{e^{2\pi i x_m t}\}$  is complete in  $L^2(S)$ , one has  $f(t + a) = 0$  on  $S$  or  $f(t) = 0$  on  $S + a$ . This proves the claim.

Now to show the completeness of  $\{g_{mn}\}$  in  $L^2(R)$ , assumes  $f \in L^2(R)$  and  $\langle f, g_{mn} \rangle = 0$ . Then

$$\langle f(t) \overline{g(t - y_n)}, e^{2\pi i x_m t} \rangle_{S+y_n} = 0$$

due to  $\text{supp } g = S$ . Because of the above claim and  $f(t) \overline{g(t - y_n)} \in L^2(S + y_n)$ , one obtains

$$f(t) \overline{g(t - y_n)} = 0$$

on  $S + y_n$  for each  $n$ . It then follows  $f(t) = 0$  on  $S + y_n$  from the assumption  $\text{supp } g = S$ . By the given condition  $\cup_n (S + y_n) = R$  almost everywhere, one has  $f = 0$  as an  $L^2$  function. This completes the proof.

**Remark:** If  $S$  is an interval, then there are several theorems to guarantee the completeness of  $\{e^{2\pi i x_m t}\}$  in  $L^2(S)$  under some assumptions of  $\{x_m\}$  ([11]). This lemma can also be compared with Theorem 1.2, since  $\cup_n (S + y_n) = R$  is equivalent to

$$0 < \sum_n |g(t + y_n)|^2.$$

To prove the main theorem in this section, we need a known result ([6]) which says that if  $\{e^{2\pi i x_m t}\}$  is a frame of  $L^2[\beta, \gamma]$ , then  $D^-(\{x_m\}) \geq \gamma - \beta$  and  $\{x_m\}$  is relatively separated. By Theorem 1.5, this can be rewritten as

**Lemma 2.5** *Let  $\{e^{2\pi i x_m t}\}$  be a frame of  $L^2[\beta, \gamma]$ . Then  $\gamma - \beta \leq D^-(X) \leq D^+(X) < +\infty$  with  $X = \{x_m\}$ .*

Recalling Condition G in Definition 1.1, we can state

**Theorem 2.2** *Let  $\{e^{2\pi i x_m t} g(t - y_n)\}$  be a Gabor basis with Condition  $G$ . Then  $D(X) = \gamma - \beta$  and  $D(Y) = 1/(\gamma - \beta)$ , where  $X = \{x_m\}$  and  $Y = \{y_n\}$ .*

Proof: Since  $\{e^{2\pi i x_m t} g(t - y_n)\}$  is a Gabor basis, one knows it is complete in  $L^2(R)$ . It follows

$$\cup_n([\beta, \gamma] + y_n) = R$$

from Lemma 2.4 and  $\text{supp } g \subseteq [\beta, \gamma]$ . Furthermore it is easy to see  $y_{n+1} - y_n \leq \gamma - \beta$ , which means  $D^-(Y) \geq 1/(\gamma - \beta)$ . But the given condition implies  $D^-(X) = D^+(X)$ ,  $D^-(Y) = D^+(Y)$  and  $D(X)D(Y) = 1$  by Theorem 2.1. Altogether one has  $D(X) \leq \gamma - \beta$ . On the other hand  $D(X) \geq \gamma - \beta$  thanks to Condition  $G$  and Lemma 2.5. Finally it follows

$$D(X) = \gamma - \beta$$

and

$$D(Y) = 1/(\gamma - \beta).$$

### 3 A Strong Balian-Low Theorem

We shall show a strong Balian-Low theorem for Gabor bases with Condition  $G$  by using Theorem 2.2 and discuss more general problem. The key step is to show next lemma.

**Lemma 3.1** *Let  $\{e^{2\pi i x_m t} g(t - y_n)\}$  be a Gabor frame with Condition  $G$ . If  $g$  is continuous on the real line  $R$ , then  $\sup_n(y_{n+1} - y_n) < \gamma - \beta$ .*

Proof: Since  $\{e^{2\pi i x_m t} g(t - y_n)\}$  is a Gabor frame, one receives  $D^+(Y) < +\infty$  by Lemma 2.3. Furthermore one can conclude that there exists some positive integer  $M$  such that at most  $M$  elements of  $Y$  are inside each interval of length  $\gamma - \beta$  due to Theorem 1.5. Now it follows

$$G(t) =: \sum_n |g(t - y_n)|^2 = \sum_{n=n_t}^{n_t+M} |g(t - y_n)|^2$$

from  $\text{supp } g \subseteq [\beta, \gamma]$ , where  $n_t$  is an integer dependent of  $t$  and  $g(t - y_{n_t}) \cdot g(t - y_{n_t+M}) \neq 0$ . It is easy to show that  $G(t)$  is continuous:

In fact given  $t_0 \in R$ ,  $G(t_0) = \sum_{n=n_0}^{n_0+M} |g(t_0 - y_n)|^2$ . Since  $g(t_0 - y_{n_0}) \neq 0$  and  $g$  is continuous, one can conclude  $g(t - y_{n_0}) \neq 0$  when  $t - t_0$  is small enough. Hence  $n_t \leq n_0$ . Similarly  $n_t + M \geq n_0 + M$ . Altogether one has  $n_t = n_0$  if  $|t - t_0|$  is small enough. Finally it follows that  $G(t)$  is continuous from the continuity of  $g$ .

Combing the continuity of the function  $G(t)$  with Theorem 1.2', one may conclude

$$0 < c \leq G(t) \leq C \tag{3.6}$$

for each  $t \in R$ . As seen in the proof of Theorem 2.2, the completeness of  $\{e^{2\pi i x_m t} g(t - y_n)\}$  implies  $\sup(y_{n+1} - y_n) \leq \gamma - \beta$ . Furthermore one can use contradiction to show the desired conclusion:

$$\sup_n (y_{n+1} - y_n) < \gamma - \beta.$$

Suppose  $\sup_n (y_{n+1} - y_n) = \gamma - \beta$ , then there would exist some subsequence  $\{y_{n_k}\}$  of  $\{y_n\}$  such that

$$\gamma - \beta - 1/k \leq y_{n_k} - y_{n_k-1}$$

for each positive integer  $k$ . It then follows that for each  $n \leq n_k - 1$ ,  $y_{n_k} + \beta - y_n \geq \gamma - 1/k$  from lemma 2.3. On the other hand,  $y_{n_k} + \beta - y_n \leq \beta$  for  $n \geq n_k$  and therefore  $g(y_{n_k} + \beta - y_n) = 0$  for  $n \geq n_k$ . Hence  $\lim_k g(y_{n_k} + \beta - y_n) = 0$  uniformly in  $\{y_n\}$ . Since  $G(t)$  is a summation of  $M$  terms, one receives that  $G(t)$  has no lower bound. This contradicts with (3.6).

Now we are ready to state the main theorem in this paper:

**Theorem 3.1** *Let  $\{e^{2\pi i x_m t} g(t - y_n)\}$  be a Gabor basis with Condition  $G$ . Then  $g(t)$  can not be continuous on the real line.*

Proof: Suppose that  $g(t)$  were continuous. Then one would have  $\sup_n (y_{n+1} - y_n) < \gamma - \beta$  by Lemma 3.1 and furthermore  $D^-(Y) > 1/(\gamma - \beta)$  with  $Y = \{y_n\}$ . On the other hand, one receives  $D^-(Y) = \frac{1}{\gamma - \beta}$  due to Theorem 2.2. A contradiction follows.

**Corollary 3.2** *Let  $\{e^{2\pi i x_m t} g(t - y_n)\}$  be a Gabor basis with Condition  $G$ . Then*

$$\|tg(t)\| \cdot \|\omega \hat{g}(\omega)\| = +\infty.$$

Proof: Note that  $g$  has compact support due to Condition  $G$  and  $\{e^{2\pi i x_m t} g(t - y_n)\}$  is a Gabor basis, one can conclude  $0 < \|tg(t)\| < +\infty$ . Suppose to the contrary that  $\|\omega \hat{g}(\omega)\| < +\infty$ . Then one would have

$$\int (\omega^2 + 1) |\hat{g}(\omega)|^2 d\omega < +\infty$$

because of  $g \in L^2(R)$ . Furthermore

$$[\int |\hat{g}(\omega)| d\omega]^2 \leq [\int (\omega^2 + 1)^{-1} d\omega] [\int (\omega^2 + 1) |\hat{g}(\omega)|^2 d\omega] < +\infty.$$

This means  $\hat{g} \in L^1(R)$  and therefore  $g$  is continuous, which contradicts with Theorem 3.1.

Theorem 3.1 and this above Corollary tells us a strong Balian-Low theorem holds for a family of non-uniform Gabor frames with Condition  $G$ . We conjecture that Condition  $G$  should be weakened.

**Conjecture:** Let  $g \in L^2(R)$  have compact support and  $\{e^{2\pi i x_m t} g(t - y_n)\}$  be a Gabor basis. Then  $g(t)$  can not be continuous.

This conjecture is at least true for uniform Gabor bases (see Theorem 3.3 below). To show that, we introduce a lemma first ([3]).

**Lemma 3.2** *Let  $g \in L^2(\mathbb{R})$ . Then  $\{e^{2\pi imt}g(t-n)\}$  forms a Gabor frame if and only if  $0 < m \leq |(Zg)(t,v)| \leq M$  holds almost everywhere for  $(t,v) \in [0,1]^2$ , where  $Z$  is the Zak transform and defined by*

$$(Zg)(t,v) =: \sum_k g(t+k)e^{2\pi ikv}.$$

By using this lemma, we give a partial solution to this above conjecture.

**Theorem 3.3** *Let  $g \in L^2(\mathbb{R})$  be continuous and have compact support. Then  $\{e^{2\pi imbt}g(t-na)\}$  never forms a Gabor basis.*

Proof: It is known that  $\{e^{2\pi imbt}g(t-na)\}$  is not a Gabor basis when  $ab \neq 1$ . Hence  $ab = 1$  is always assumed in this proof. Firstly one shows this theorem in the case  $a = b = 1$  and  $\text{supp } g \subseteq [0, N]$  for some positive integer  $N$ .

Note that the Zak transform  $(Zg)(t,v) =: \sum_k g(t+k)e^{2\pi ikv}$  is continuous in  $\mathbb{R}^2$ , since  $g$  is compactly supported and continuous. Furthermore the assumption  $\text{supp } g \subseteq [0, N]$  implies

$$(Zg)(t,v) = \sum_{k=0}^{N-1} g(t+k)e^{2\pi ikv}$$

for  $0 \leq t \leq 1$ . Define  $G(t) =: (Zg)(t,0)(Zg)(t,1/2) = [\sum_{k=0}^{N-1} g(t+k)][\sum_{k=0}^{N-1} (-1)^k g(t+k)]$ . Then  $G(t)$  is continuous and

$$G(0)G(1) = -[\sum_{k=1}^N g(k)]^2 [\sum_{k=1}^{N-1} (-1)^k g(k)]^2,$$

where the fact  $g(0) = g(N) = 0$  is used. It is clear that  $G(0)G(1) \leq 0$ . Furthermore by mean-valued theorem, there exists some  $0 \leq t_0 \leq 1$  such that  $G(t_0) = 0$ . That is, either  $(Zg)(t_0,0) = 0$  or  $(Zg)(t_0,1/2) = 0$ . But  $(Zg)(t,v)$  is continuous as mentioned above, one can conclude  $|(Zg)(t,v)|$  doesn't have lower bound, which implies  $\{e^{2\pi imt}g(t-n)\}$  is not a frame by Lemma 3.2.

Next one proves this theorem under the assumptions  $a = b = 1$  and  $\text{supp } g \subseteq [\beta, \gamma]$  for any real numbers  $\beta < \gamma$ . Suppose  $\{e^{2\pi imt}g(t-n)\}$  were Gabor basis, then  $\{e^{-2\pi im\beta}e^{2\pi imt}g_0(t-n)\}$  would be as well with

$$g_0(t) = g(t-\beta),$$

since the translation operator is unitary. Because  $\{e^{-2\pi im\beta}\}$  is a sequence of constants with moduli 1, one can show that  $\{e^{2\pi imt}g_0(t-n)\}$  is a Gabor basis. The continuity of  $g_0$  follows from that of  $g$ . This contradicts with the first part.

Finally one need prove the theorem for the most general case:  $ab = 1$  and  $\text{supp } g \subseteq [\beta, \gamma]$ . Suppose  $\{e^{2\pi imbt}g(t-na)\}$  were a Gabor basis. Then  $\{e^{2\pi imt}(Tg)(t-n)\}$  would be as well, where

$$(Tf)(t) =: \sqrt{a}f(at)$$

is an unitary operator. It is easy to see that  $Tg \in L^2(R)$  has compact support and is continuous from the corresponding properties of  $g$ . Now a contradiction follows from the second case.

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## THE SPECTRUM OF FUNCTIONS DEFINED ON $\mathbb{R}^+$ ; APPLICATION TO FUNCTIONAL EQUATIONS

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ABSTRACT. Suppose that  $M$  is a closed subspace of  $L^\infty(\mathbb{R}^+, X)$ , where  $X$  is a complex Banach space. We define the  $M$ -spectrum  $\sigma_M(u)$  of a function  $u \in L^\infty(\mathbb{R}^+, X)$ . Certain conditions will be supposed on both  $M$  and  $\sigma_M(u)$  to insure the existence of  $u \in M$ . We prove that if  $u$  is uniformly continuous and bounded, such that  $\sigma_M(u)$  is at most countable and, for every  $\lambda \in \sigma_M(u)$ , the function  $e^{-i\lambda t}u(t)$  is ergodic, then  $u \in M$ . We apply this result to the integro-differential operator equation

$$\alpha(u'(t) - Au(t)) + \beta \int_0^t u(t-s) d\mu(s) = f(t), \quad t \geq 0,$$

where the free term  $f \in M \cap C_{ub}(\mathbb{R}^+, X)$  and  $\alpha, \beta \in \mathbb{C}$ . Here,  $A$  is the generator of a  $C_0$ -semigroup of linear bounded operators defined on  $X$ ,  $D(A)$  is the domain of  $A$ , and  $\mu$  is a bounded Borel measure on  $\mathbb{R}^+$ . Certain conditions will be imposed to guarantee the existence of solutions in the class  $M$ .

### I. Introduction

Ruess and Summers [9], [10] considered the homogeneous abstract Cauchy problem

$$\begin{aligned} u'(t) - Au(t) &= 0, \quad t > 0, \\ u(0) &= x_0 \end{aligned} \tag{I.1}$$

associated with a generator  $A$  of a uniformly bounded  $C_0$ -semigroup of linear bounded operators  $(T(t))_{\{t \geq 0\}}$  on  $X$ . They proved that, for  $x_0 \in D(A)$ , the (unique) solution  $u(t) = T(t)x_0$  of (I.1) is asymptotically almost periodic (a.a.p.) (Eberlein weakly almost periodic (w.a.p.-E)), provided that it has relatively compact (weakly relatively compact) range. Here,  $D(A)$  is domain of  $A$ . In particular, when, in addition to the above assumptions,  $X$  is a reflexive Banach space, all solutions of (I.1) are w.a.p.-E. A function  $f \in C_b(\mathbb{R}^+, X)$  is said to be a.a.p. (w.a.p.-E) if the set of translates  $\{f_\omega : \omega \in \mathbb{R}^+\}$  is relatively compact (weakly relatively

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compact) in  $C_b(\mathbb{R}^+, X)$ . A translate  $f_\omega$  is defined by  $f_\omega(t) = f(t + \omega)$ . Here  $C_b(\mathbb{R}^+, X)$  is the space of all continuous bounded functions from  $\mathbb{R}^+$  to  $X$ . The space of all a.a.p. functions is denoted by  $AAP(\mathbb{R}^+, X)$  and the space of all w.a.p.-E is denoted by  $W(\mathbb{R}^+, X)$ . It is well-known that  $W(\mathbb{R}^+, X) \subset C_{ub}(\mathbb{R}^+, X)$  the space of all uniformly continuous bounded functions, see [11]. Phong and Lyubich [12] proved that, assuming that  $X$  is reflexive, if  $SP(A) \cap i\mathbb{R}$  is at most countable, then all solutions of (I.1) are a.a.p. The spectrum  $SP(A)$  of  $A$  is defined by

$$SP(A) = \{\lambda \in \mathbb{C} : A - \lambda I \text{ has no bounded inverse on } X\}.$$

Throughout the paper,  $X$  is a complex Banach space with the norm  $\| \cdot \|$ . As usual  $L^\infty(\mathbb{R}^+, X)$  denotes the Banach space of all essentially bounded measurable functions defined on  $\mathbb{R}^+$  with the norm  $\|f\|_\infty = \text{ess sup}_{t \in \mathbb{R}^+} \|f(t)\|$ . A function  $f$  is called measurable if there exists a sequence of simple functions  $\{f_n\}$  such that  $f_n \rightarrow f$  a.e. with respect to the Lebesgue measure  $m$ . By a simple function it is meant a function of the form  $\sum_{i=1}^n x_i \chi_{A_i}$ ,  $x_i \in X$  and  $\chi_{A_i}$  is the characteristic function of the Lebesgue measurable set  $A_i$  with finite measure. Finally,  $M$  denotes a closed subspace of  $L^\infty(\mathbb{R}^+, X)$  satisfying at least one of the following conditions :

**(P1)** For any  $\phi \in C_{ub}(\mathbb{R}, X)$ , if  $\phi|_{\mathbb{R}^+} \in M$  then  $\phi_s|_{\mathbb{R}^+} \in M$  for every  $s \in \mathbb{R}$ .

**(P2)**  $M$  contains the constant functions.

**(P3)**  $M$  is invariant under multiplication by characters, i.e.  $\check{\lambda} f \in M$  for any  $f \in M$  and  $\lambda \in \mathbb{R}$ , where

$$\check{\lambda}(t) = e^{i\lambda t}.$$

**(P4)**  $Au \in M$  for any  $A \in B(X)$  and  $u \in M$ ,

where  $B(X)$  is the space of all linear bounded operators on  $X$ . At first we recall some spaces which satisfy conditions **(P1)**-**(P4)**. A continuous function  $f$  from  $\mathbb{R}$  to  $X$  is called almost periodic (a.p.) if the set of translates  $\{f_\omega : \omega \in \mathbb{R}\}$  is relatively compact in  $C_b(\mathbb{R}, X)$  the space of all continuous bounded functions from  $\mathbb{R}$  to  $X$ . The space of all a.p. functions is denoted by  $AP(\mathbb{R}, X)$ . We denote by  $AP(\mathbb{R}^+, X) = AP(\mathbb{R}, X)|_{\mathbb{R}^+}$  the restriction of a.p. functions on  $\mathbb{R}^+$ . A function  $f \in C_b(\mathbb{R}, X)$  is said to be almost automorphic (a.a.) if for each sequence  $\{a'_n\} \subset \mathbb{R}$ , there exists a subsequence  $\{a_n\}$  such that

(i)  $\lim_{n \rightarrow \infty} f(t + a_n) = g(t)$ ,  $t \in \mathbb{R}$ , where  $g$  is a continuous function.

(ii)  $\lim_{n \rightarrow \infty} g(t - a_n) = f(t)$ ,  $t \in \mathbb{R}$ .

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It is well known that an a.a. function is uniformly continuous and its range is totally bounded. The space of all a.a. functions is denoted by  $AA(\mathbb{R}, X)$ . We denote by  $AA(\mathbb{R}^+, X) = AA(\mathbb{R}, X)|_{\mathbb{R}^+}$  the restrictions of a.a. functions on  $\mathbb{R}^+$ . A continuous function  $f$  from  $\mathbb{R}^+$  to  $X$  is called asymptotically almost automorphic (a.a.a.) if it can be written in the form  $f = p + q$  where  $p \in AA(\mathbb{R}^+, X)$  and  $q \in C_0(\mathbb{R}^+, X)$  the space of all continuous functions on  $\mathbb{R}^+$  vanishing at  $\infty$ . The space of all a.a.a. functions is denoted by  $AAA(\mathbb{R}^+, X)$ . A function  $f \in L^\infty(\mathbb{R}^+, X)$  is called a.a.p. in the sense of Staffans if it has the decomposition  $f = p + q$ , where  $p \in AP(\mathbb{R}^+, X)$  and  $q \in L^\infty(\mathbb{R}^+, X)$  vanishes at  $\infty$ . The space of all a.a.p. in the sense of Staffans is denoted by  $S - AAP(\mathbb{R}^+, X)$ . For more details and properties of these spaces and others, see [5] and [6].

In Section II the  $M$ -spectrum  $\sigma_M(u)$  of a function  $u \in L^\infty(\mathbb{R}^+, X)$  will be defined by

$$\sigma_M(u) = Z(I_M(u)) = \{\alpha \in \mathbb{R} : \hat{f}(\alpha) = 0 \ \forall f \in I_M(u)\},$$

where

$$I_M(u) = \{f \in L^1(\mathbb{R}) : f \odot u \in M\}$$

and

$$f \odot u(t) = \int_{\mathbb{R}^+} f(t-s)u(s) \, ds, \quad t \in \mathbb{R}^+.$$

Here,  $\hat{f}(\alpha) = \int_{\mathbb{R}} f(t)e^{-i\alpha t} \, dt$ . When  $M$  is a closed subspace of  $L^\infty(\mathbb{R}, X)$  and  $u \in L^\infty(\mathbb{R}, X)$ , the  $M$ -spectrum  $\sigma_M(u)$  was defined in [1] by

$$\sigma_M(u) = Z(I_M(u)), \text{ where } I_M(u) = \{f \in L^1(\mathbb{R}) : f * u \in M\}.$$

We need the following Lemma [5, Lemma 4.1.2] in proving the main results of this section

**LEMMA 1.0.**

*Let  $\mathfrak{S}$  be a closed subspace of  $L^\infty(\mathbb{R}, X)$ . If  $\mathfrak{S}$  is translation invariant, i.e. it satisfies the following condition*

$$u_s \in \mathfrak{S} \quad \text{for any } u \in \mathfrak{S} \quad \text{and any } s \in \mathbb{R},$$

*then  $f * u \in \mathfrak{S}$  for any  $f \in L^1(\mathbb{R})$  and any  $u \in \mathfrak{S} \cap C_{ub}(\mathbb{R}, X)$ .*

*Proof.* Let  $f \in L^1(\mathbb{R})$  and  $u \in \mathfrak{S} \cap C_{ub}(\mathbb{R}, X)$ . Define the function  $g : \mathbb{R} \rightarrow \mathfrak{S}$  by

$$g(s) = u_{-s}.$$

The function  $g$  is continuous and bounded, since  $u$  is uniformly continuous. Applying Bochner's Theorem [13, p. 133], we get  $\int_{\mathbb{R}} f(s)u_{-s} ds \in \mathfrak{S}$ , whence  $f * u \in \mathfrak{S}$

We show in this section the following similar result :

If  $M$  is a closed subspace of  $L^\infty(\mathbb{R}^+, X)$  satisfying **(P1)**, then

$$f \odot u \in M \quad \text{for any } f \in L^1(\mathbb{R}) \text{ and any } u \in M \bigcap C_{ub}(\mathbb{R}^+, X).$$

Also we prove Theorem II.8 which includes some needed properties of  $\sigma_M(u)$ .

In Section III we prove similar results as in [5] for functions defined on  $\mathbb{R}^+$ . We show the following result [6, Theorem III.3.2]:

### THEOREM I.1.

*Let  $u \in C_{ub}(\mathbb{R}^+, X)$ . If  $\sigma_M(u)$  is at most countable such that the function  $(-\lambda)\check{u}$  is ergodic for every  $\lambda \in \sigma_M(u)$ , then  $u \in M$ .*

We recall a function  $u \in L^\infty(\mathbb{R}^+, X)$  is called ergodic if there exists  $x \in X$  such that

$$\lim_{T \rightarrow \infty} \|1/T \int_0^T (u_s - x) ds\|_\infty = 0.$$

The space of all ergodic functions is denoted by  $E(\mathbb{R}^+, X)$ . A function  $u \in L^\infty(\mathbb{R}^+, X)$  is called totally ergodic if  $\check{\lambda}u$  is ergodic for every  $\lambda \in \mathbb{R}$ . The space of all totally ergodic functions is denoted by  $TE(\mathbb{R}^+, X)$ . The fact that  $W(\mathbb{R}^+, X) \subset TE(\mathbb{R}^+, X)$  is a result of [4]. Theorem I.1 plays an essential role in proving the existence of solutions in the class  $M$ . It can be applied to the equations

$$\begin{aligned} u'(t) - Au(t) &= f(t), \quad t > 0, \\ u(0) &= x_0 \in D(A) \end{aligned} \tag{I.2}$$

and

$$\int_{-\infty}^t u(t-s) d\mu(s) = f(t), \quad t \in \mathbb{R}^+ \tag{I.3}$$

$$\int_0^t u(t-s) d\nu(s) = f(t), \quad t \in \mathbb{R}^+ \tag{I.4}$$

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to obtain conditions that insure the existence of solutions in  $M$ , where the free term  $f \in M \cap C_{ub}(\mathbb{R}^+, X)$ ,  $\mu$  is a bounded Borel measure on  $\mathbb{R}$ , and  $\nu$  is a bounded Borel measure on  $\mathbb{R}^+$ .

Section IV is devoted to the integro-differential operator equation

$$\alpha(u'(t) - Au(t)) + \beta \int_0^t u(t-s) d\mu(s) = f(t), \quad t \geq 0, \quad (\text{I.5})$$

where the free term  $f \in M \cap C_{ub}(\mathbb{R}^+, X)$  and  $\alpha, \beta \in \mathbb{C}$ . Here  $\mu$  is a bounded Borel measure on  $\mathbb{R}^+$  and  $A$  is the generator of a  $C_0$ -semigroup of linear bounded operators  $(T(t))_{t \geq 0}$  defined on  $X$ . We denote by

$$SP(A, \mu) = \{\lambda \in \mathbb{R} : \alpha A - (\alpha i\lambda + \beta \hat{\mu}(\lambda))I \text{ has no bounded inverse on } X\}.$$

Here,  $\hat{\mu}(\lambda) = \int_{\mathbb{R}^+} e^{-i\lambda t} d\mu(t)$ . Our aim of this paper is to prove that if  $SP(A, \mu)$  is at most countable, then every solution  $u$  of equation (I.5) belongs to  $M$ , provided that the function  $e^{-i\lambda t}u(t)$  is ergodic for every  $\lambda \in SP(A, \mu)$ .

When  $\alpha = 1$  and  $\beta = 0$ , we get the following result [6, Theorem III.4.2]

**THEOREM I.2.** *Let  $M$  be a Banach subspace of  $L^\infty(\mathbb{R}^+, X)$  that satisfies (P1)-(P4). Suppose that  $SP(A) \cap i\mathbb{R}$  is at most countable. If  $u \in C_{ub}(\mathbb{R}^+, X)$  is the mild solution of (I.2), such that for every  $\lambda \in SP(A) \cap i\mathbb{R}$ , the function  $e^{-\lambda t}u(t)$  is ergodic, then  $u \in M$ .*

When  $\alpha = 0$  and  $\beta = 1$ , we obtain the following result [6, Theorem III.5.1]

**THEOREM I.3.** *Suppose that  $M$  is a closed subspace of  $L^\infty(\mathbb{R}^+, X)$  satisfying (P1)-(P3). If  $Z(\mu) = \{\alpha \in \mathbb{R} : \hat{\mu}(\alpha) = 0\}$  is at most countable, then every solution  $u \in C_{ub}(\mathbb{R}^+, X)$  of equation (I.3) belongs to  $M$ , provided that the function  $e^{-i\alpha t}u(t)$  is ergodic for every  $\alpha \in Z(\mu)$ .*

## II. The $M$ -spectrum of functions in $L^\infty(\mathbb{R}^+, X)$

In this section we define the  $M$ -spectrum  $\sigma_M(u)$  of a function  $u \in L^\infty(\mathbb{R}^+, X)$ . We prove some properties of this spectrum in order to arrive to the main result. For a function  $u \in L^\infty(\mathbb{R}, X)$  and a closed subspace  $M$  of  $L^\infty(\mathbb{R}, X)$ , the  $M$ -spectrum was defined according to the definition given in the Introduction, see [5] and [1].

**Definition II.1.**

(i) We define the operation  $\odot : L^1(\mathbb{R}) \times L^\infty(\mathbb{R}^+, X) \rightarrow L^\infty(\mathbb{R}^+, X)$  by

$$(f, u) \mapsto f \odot u, \text{ where}$$

$$f \odot u(t) = \int_{\mathbb{R}^+} f(t-s)u(s) ds, \quad t \in \mathbb{R}^+.$$

(ii) For a function  $u \in L^\infty(\mathbb{R}^+, X)$ , we define  $\tilde{u}$  on  $\mathbb{R}$  by

$$\tilde{u}(t) = \begin{cases} u(t), & \text{if } t \geq 0 \\ 0, & \text{if } t < 0. \end{cases}$$

It is clear that  $f \odot u = (f * \tilde{u})|_{\mathbb{R}^+}$ ,  $f \in L^1(\mathbb{R})$ , where  $*$  is the ordinary convolution.

**Lemma II.2.**

*If  $M$  is a closed subspace of  $L^\infty(\mathbb{R}^+, X)$  satisfying (P1), then  $M$  contains  $C_0(\mathbb{R}^+, X)$  the space of all continuous functions on  $\mathbb{R}^+$  vanishing at  $\infty$ .*

*Consequently, if in addition  $M$  satisfies (P2), then  $M$  contains the space of all continuous functions on  $\mathbb{R}^+$  having limit at  $\infty$ .*

*Proof.* It is enough to show that  $C_c(\mathbb{R}^+, X)$  the space of all continuous functions with compact support is a subset of  $M$ . This is because  $C_c(\mathbb{R}^+, X)$  is dense in  $C_0(\mathbb{R}^+, X)$ . Let  $f \in C_c(\mathbb{R}^+, X)$  with  $f(t) = 0$  for all  $t \geq T$  for some  $T > 0$ . Define  $F$  on  $\mathbb{R}$  by

$$F(t) = \begin{cases} f(t), & \text{if } t \geq 0 \\ f(0), & \text{if } t < 0. \end{cases}$$

The function  $G = F_T$  belongs to  $C_{ub}(\mathbb{R}, X)$  and  $G|_{\mathbb{R}^+} = 0 \in M$ . Condition (P1) implies that  $G_{-T}|_{\mathbb{R}^+} \in M$  whence  $f \in M$ .

In the following, for a closed subspace  $M$  of  $L^\infty(\mathbb{R}^+, X)$ , we denote by

$$\mathfrak{S} = \{\phi \in C_{ub}(\mathbb{R}, X) : \phi|_{\mathbb{R}^+} \in M\}.$$

**Lemma II.3.**

- (i) *If  $M$  is a closed subspace of  $L^\infty(\mathbb{R}^+, X)$  satisfying (P1), then  $\mathfrak{S}$  is a closed subspace of  $L^\infty(\mathbb{R}, X)$  which is translation invariant.*
- (ii) *If  $M$  satisfies (P2)((P3)), then so is  $\mathfrak{S}$ .*

*Proof.*

- (i) Let  $M$  be a closed subspace of  $L^\infty(\mathbb{R}^+, X)$  satisfying **(P1)**. Let  $\{f_n\}$  be a sequence in  $\mathfrak{S}$  such that  $\|f_n - f\|_\infty \rightarrow 0$  as  $n \rightarrow \infty$  for some  $f \in C_{ub}(\mathbb{R}, X)$ . Since  $f_n|_{\mathbb{R}^+} \in M$ , then  $f|_{\mathbb{R}^+} \in M$  and  $\mathfrak{S}$  is a closed subspace. The condition **(P1)** implies that  $\mathfrak{S}$  is translation invariant.
- (ii) Assume that  $M$  satisfies **(P2)**. The restriction of a constant function  $x : \mathbb{R} \rightarrow X$ , which is uniformly continuous, on  $\mathbb{R}^+$  belongs to  $M$ . Then  $x \in \mathfrak{S}$  and  $\mathfrak{S}$  satisfies **(P2)**. Assume that  $M$  satisfies **(P3)**. Let  $\lambda \in \mathbb{R}$  and  $f \in \mathfrak{S}$ . Then  $\check{\lambda}f \in C_{ub}(\mathbb{R}, X)$ . Since the restriction  $\check{\lambda}f|_{\mathbb{R}^+} \in M$ , then  $\check{\lambda}f \in \mathfrak{S}$  and  $\mathfrak{S}$  satisfies **(P3)**.

**Lemma II.4.**

*If  $M$  is a closed subspace of  $L^\infty(\mathbb{R}^+, X)$  satisfying **(P1)**, then*

$$\forall f \in L^1(\mathbb{R}) \quad \forall u \in M \cap C_{ub}(\mathbb{R}^+, X) \quad (f \odot u \in M).$$

*Proof.* Let  $M$  be a closed subspace of  $L^\infty(\mathbb{R}^+, X)$  satisfying **(P1)**. Let  $f \in L^1(\mathbb{R})$  and  $u \in M \cap C_{ub}(\mathbb{R}^+, X)$ .

Define the function  $\bar{u}$  on  $\mathbb{R}$  by

$$\bar{u}(t) = \begin{cases} u(t), & \text{if } t \geq 0 \\ u(0), & \text{if } t < 0. \end{cases}$$

The function  $\bar{u}$  belongs to  $\mathfrak{S}$ . By Lemma I.0,  $f * \bar{u} \in \mathfrak{S}$  whence  $(f * \bar{u})|_{\mathbb{R}^+} \in M$ . We have

$$f * \bar{u}(t) = f \odot u(t) + \int_t^\infty f(s) ds u(0), \quad t \in \mathbb{R}^+.$$

Since  $\lim_{t \rightarrow \infty} \int_t^\infty f(s) ds u(0) = 0$ , then it belongs to  $M$ , from lemma II.2. Hence  $f \odot u \in M$ .

**Definition II.5.**

Assume that  $M$  is a closed subspace of  $L^\infty(\mathbb{R}^+, X)$  satisfying **(P1)**. Let  $u \in L^\infty(\mathbb{R}^+, X)$ . We denote by

$$I_M(u) = \{f \in L^1(\mathbb{R}) : f \odot u \in M\}.$$

We define the  $M$ -spectrum  $\sigma_M(u)$  of  $u \in L^\infty(\mathbb{R}^+, X)$  by

$$\begin{aligned} \sigma_M(u) &= Z(I_M(u)) = \{\alpha \in \mathbb{R} : \hat{f}(\alpha) = 0 \quad \forall f \in I_M(u)\} \\ &= \sigma_{\mathfrak{S}}(\tilde{u}), \end{aligned}$$

where  $\hat{f}(\alpha) = \int_{\mathbb{R}} f(t)e^{-i\alpha t} dt$  and  $\tilde{u}$  was defined in Definition II.1.

We recall

$$I_{\mathfrak{S}}(\tilde{u}) = \{f \in L^1(\mathbb{R}) : f * \tilde{u} \in \mathfrak{S}\}$$

and

$$\sigma_{\mathfrak{S}}(\tilde{u}) = \{\alpha \in \mathbb{R} : \hat{f}(\alpha) = 0 \ \forall f \in I_{\mathfrak{S}}(\tilde{u})\},$$

see [5], [1].

**Lemma II.6.**

*If  $u \in L^\infty(\mathbb{R}^+, X)$ , then the following conditions are equivalent*

- (i)  $u \in C_{ub}(\mathbb{R}^+, X)$ ,
- (ii)  $\lim_{t \rightarrow 0} \|u_t - u\|_\infty = 0$ ,
- (iii)  $\lim_{T \rightarrow 0} \|\rho_T \odot u - u\|_\infty = 0$ , where  $\rho_T = 1/T \chi_{[-T, 0]}$ ,  $T > 0$ . Here  $\chi_{[-T, 0]}$  is the characteristic function of the interval  $[-T, 0]$ .

This is a classical result in the theory of  $L^1(G)$ -modules. We can replace  $\{\rho_T\}$  by any approximate of identity (see [3]).

One can verify the following

**Lemma II.7.**

*If  $M$  is a closed subspace of  $L^\infty(\mathbb{R}^+, X)$  satisfying **(P1)**, then*

- (i) *For any  $f, h \in L^1(\mathbb{R})$  and  $u \in L^\infty(\mathbb{R}^+, X)$ , we have*

$$(h * f) \odot u - h \odot (f \odot u) \in M, \tag{II.7.1}$$

$$f \odot (h \odot u) - h \odot (f \odot u) \in M. \tag{II.7.2}$$

- (ii) *For any  $f \in L^1(\mathbb{R})$  and  $u \in L^\infty(\mathbb{R}^+, X)$  we have*

$$(f_s - f) \odot u - f \odot (u_s - u) \in M \ \forall s \in \mathbb{R}^+. \tag{II.7.3}$$



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*Proof.* In order to show (i), Let  $f, h \in L^1(\mathbb{R})$  and  $u \in L^\infty(\mathbb{R}^+, X)$ . We define  $g$  on  $\mathbb{R}$  by  $g = f * \tilde{u}$ . We have

$$h * g(t) = h \odot (f \odot u)(t) + \int_{\mathbb{R}^-} h(t-s)g(s) ds, \quad t \in \mathbb{R}^+.$$

The function  $\int_{\mathbb{R}^-} h(t-s)g(s) ds$  belongs to  $M$ , since this is true for every  $h \in C_c(\mathbb{R})$ , the space of all continuous functions on  $\mathbb{R}$  with compact support, and  $C_c(\mathbb{R})$  is dense in  $L^1(\mathbb{R})$ . Hence  $(h * f) \odot u - h \odot (f \odot u) \in M$ .

We can prove similarly that  $(f * h) \odot u - f \odot (h \odot u) \in M$ . This implies that (II.7.2) holds.

Now we show (ii). Let  $f \in L^1(\mathbb{R})$  and  $u \in L^\infty(\mathbb{R}^+, X)$ . A simple calculation shows that

$$(f_s - f) \odot u(t) = f \odot (u_s - u)(t) + \int_0^s f(t+s-y)u(y) dy, \quad s \in \mathbb{R}^+, t \in \mathbb{R}^+.$$

The function  $\int_0^s f(t+s-y)u(y) dy$  belongs to  $M$  for every  $s \in \mathbb{R}^+$ , since this is true for every  $f \in C_c(\mathbb{R})$  and  $M$  is closed.

**Theorem II.8.**

Let  $u \in L^\infty(\mathbb{R}^+, X)$ . If  $M$  is a closed subspace of  $L^\infty(\mathbb{R}^+, X)$  satisfying **(P1)**, then the following hold :

- (1)  $\sigma_M(u) = \emptyset$  iff for every  $f \in L^1(\mathbb{R})$ ,  $f \odot u \in M$ .
- (2) If  $u \in C_{ub}(\mathbb{R}^+, X)$ , then  $\sigma_M(u) = \emptyset$  iff  $u \in M$ .
- (3) If  $\sigma_M(u) = \{0\}$ , then  $f \odot (u_s - u) \in M$  for every  $f \in L^1(\mathbb{R})$  and for every  $s \in \mathbb{R}^+$ .
- (4) If  $u \in C_{ub}(\mathbb{R}^+, X)$ , then  $\sigma_M(u) = \{0\} \implies u_s - u \in M \quad \forall s \in \mathbb{R}^+$ .
- (5)  $\sigma_M(f \odot u) \subseteq \text{supp } \hat{f} \cap \sigma_M(u) \quad \forall f \in L^1(\mathbb{R})$ .
- (6) If in addition  $M$  satisfies (P3), then

$$\sigma_M(\check{\gamma}u) = \sigma_M(u) + \gamma \quad \forall \gamma \in \mathbb{R},$$

where  $\check{\gamma}(t) = e^{i\gamma t}$ .

*Proof.* (1) We have  $Z(I_M(u)) = \emptyset$  iff  $I_M(u) = L^1(\mathbb{R})$ . Hence  $\sigma_M(u) = \emptyset$  iff  $f \odot u \in M$  for every  $f \in L^1(\mathbb{R})$ .

(2) Let  $u \in C_{ub}(\mathbb{R}^+, X)$ . Suppose that  $\sigma_M(u) = \emptyset$ . Hence, by (1) we have  $f \odot u \in M$  for any  $f \in L^1(\mathbb{R})$ . By Lemma II.6,  $\lim_{T \rightarrow 0} \|\rho_T \odot u - u\|_\infty = 0$  whence  $u \in M$ . Conversely, suppose that  $u \in M$ . By Lemma II.4, we get that  $f \odot u \in M$  for every  $f \in L^1(\mathbb{R})$ , which in return implies that  $\sigma_M(u) = \emptyset$ .

(3) Suppose that  $\sigma_M(u) = \{0\}$  i.e  $Z(I_M(u)) = \{0\}$ . We can see that  $I_M(u) = \{f \in L^1(\mathbb{R}) : \hat{f}(0) = 0\}$ . ( $\{0\}$  is a set of spectral synthesis ). Hence,  $(f_s - f) \odot u \in M$  for every  $s \in \mathbb{R}^+$  and every  $f \in L^1(\mathbb{R})$  whence

$f \odot (u_s - u) \in M$  for any  $s \in \mathbb{R}^+$  and any  $f \in L^1(\mathbb{R})$ , from (II.7.3).

(4) is a direct consequence of (3) and (2).

(5) Let  $f \in L^1(\mathbb{R})$ . Let  $\alpha \in \sigma_M(f \odot u)$ . To show that  $\alpha \in \sigma_M(u)$ , let  $h \in L^1(\mathbb{R})$  be such that  $h \odot u \in M$ . We have, by II.4,  $f \odot (h \odot u) \in M$ . Hence  $h \odot (f \odot u) \in M$ , this is from (II.7.2) whence  $\hat{h}(\alpha) = 0$ . Now we show that  $\alpha \in \text{supp } \hat{f}$ . Suppose on the contrary that  $\alpha \notin \text{supp } \hat{f}$ . Then there exists  $g \in L^1(\mathbb{R})$  such that  $\hat{g}(\alpha) \neq 0$  and  $\hat{g}(\text{supp } \hat{f}) = \{0\}$ . We have  $g * f = 0$  whence  $(g * f) \odot u = 0 \in M$ . By (II.7.1), we get  $g \odot (f \odot u) \in M$  and  $\hat{g}(\alpha) = 0$  which is a contradiction.

(6) We denote by  $g = \check{\gamma}_0 u$ ,  $\gamma_0 \in \mathbb{R}$ . Let  $\gamma \in \sigma_M(g)$ . Let  $f \in L^1(\mathbb{R})$  be such that  $f \odot u \in M$ . A simple calculation shows that

$$(\check{\gamma}_0 f) \odot g = \check{\gamma}_0 (f \odot u).$$

Hence,  $(\check{\gamma}_0 f) \odot g \in M$  whence  $(\check{\gamma}_0 f)^\wedge(\gamma) = 0$  i.e.  $\hat{f}(\gamma - \gamma_0) = 0$  and we get  $\gamma - \gamma_0 \in \sigma_M(u)$ . Conversely, let  $\gamma \in \sigma_M(u)$  and  $f \in L^1(\mathbb{R})$  be such that  $f \odot g \in M$ . We have

$$f \odot g = \check{\gamma}_0 [((- \gamma_0)^\check{f}) \odot u].$$

Hence,  $((- \gamma_0)^\check{f}) \odot u \in M$  whence  $((- \gamma_0)^\check{f})^\wedge(\gamma) = 0$  i.e.  $\hat{f}(\gamma + \gamma_0) = 0$  and we get  $\gamma + \gamma_0 \in \sigma_M(g)$ .

### III. Spectral characterization of the classes $M$

Let  $M$  be a closed subspace of  $L^\infty(\mathbb{R}^+, X)$ . Assuming that  $M$  satisfies **(P1)**-**(P3)**, we prove the following result : if  $\phi \in C_{ub}(\mathbb{R}^+, X)$  is such that the  $M$ -spectrum  $\sigma_M(\phi)$  of  $\phi$  is at most countable and, for every  $\lambda \in \sigma_M(\phi)$ , the function  $e^{-i\lambda t}\phi(t)$  is ergodic, then  $\phi \in M$ .

#### Lemma III.1.

*If  $\lambda_0 \in \mathbb{R}$  is such that  $(-\lambda_0)^\check{\phi} \in E(\mathbb{R}^+, X) \cap C_{ub}(\mathbb{R}^+, X)$ , then  $\lambda_0$  cannot be an isolated point of  $\sigma_M(\phi)$ .*

*Proof.* Let  $\lambda_0 \in \mathbb{R}$  be such that  $(-\lambda_0)^\check{\phi} \in E(\mathbb{R}^+, X) \cap C_{ub}(\mathbb{R}^+, X)$ . Suppose on the contrary that  $\lambda_0$  is an isolated point of  $\sigma_M(\phi)$ . There exists a compact neighbourhood  $V$  of  $\lambda_0$  such that  $V \cap (\sigma_M(\phi) \setminus \{\lambda_0\}) = \emptyset$ . Choose  $f \in L^1(\mathbb{R})$  such that  $\hat{f}(\lambda_0) \neq 0$  and  $\hat{f}(\mathbb{C}V) = \{0\}$ . Here,  $\mathbb{C}V$  is the complement of  $V$ . Hence,  $\sigma_M(f \odot \phi) \subseteq \sigma_M(\phi) \cap \text{supp } \hat{f} \subseteq \{\lambda_0\}$  whence  $\sigma_M(f \odot \phi) = \{\lambda_0\}$ . By theorem II.8, we get

$$[(-\lambda_0)^\check{(f \odot \phi)}]_s = [(-\lambda_0)^\check{(f \odot \phi)}] \in M \quad \forall s \in \mathbb{R}^+.$$

Since  $(-\lambda_0)^\check{(f \odot \phi)} = (-\lambda_0)^\check{f} \odot (-\lambda_0)^\check{\phi} \in E(\mathbb{R}^+, X) \cap C_{ub}(\mathbb{R}^+, X)$ , because of lemma II.4, then we get  $(-\lambda_0)^\check{(f \odot \phi)} \in M$  whence  $f \odot \phi \in M$ . Hence,  $\hat{f}(\lambda_0) = 0$  which is a contradiction.

**Theorem III.2.**

Let  $\phi \in C_{ub}(\mathbb{R}^+, X)$ . If  $\sigma_M(\phi)$  is at most countable such that the function  $(-\lambda)\check{\phi} \in E(\mathbb{R}^+, X)$ , for every  $\lambda \in \sigma_M(\phi)$ , then  $\phi \in M$ .

*Proof.* Suppose that  $\phi \in C_{ub}(\mathbb{R}^+, X)$  satisfying the hypothesis of the theorem. We show that  $\sigma_M(\phi) = \emptyset$ . Suppose on the contrary that  $\sigma_M(\phi) \neq \emptyset$ . Then  $\sigma_M(\phi)$  has an isolated point  $\lambda_0$  [2]. Since  $(-\lambda_0)\check{\phi} \in E(\mathbb{R}^+, X) \cap C_{ub}(\mathbb{R}^+, X)$ , then by Lemma III.1, we get that  $\lambda_0$  is not an isolated point of  $\sigma_M(\phi)$  which is a contradiction.

**Corollary III.3.**

Let  $\phi \in C_{ub}(\mathbb{R}^+, X) \cap TE(\mathbb{R}^+, X)$ . If  $\sigma_M(\phi)$  is at most countable, then  $\phi \in M$ .

**IV. An application to integro-differential operator equations**

In this section we consider the equation

$$\alpha(u'(t) - Au(t)) + \beta \int_0^t u(t-s) d\mu(s) = f(t), \quad t \geq 0 \quad (\text{IV.1})$$

where the free term  $f \in M \cap C_{ub}(\mathbb{R}^+, X)$  and  $\alpha, \beta \in \mathbb{C}, \alpha \neq 0$ . Here  $\mu$  is a bounded Borel measure on  $\mathbb{R}^+$  and  $A$  is the generator of a  $C_0$ -semigroup of linear bounded operators  $(T(t))_{t \geq 0}$  defined on  $X$ . We denote by

$$SP(A, \mu) = \{\lambda \in \mathbb{R} : \alpha A - (\alpha i\lambda + \beta \hat{\mu}(\lambda))I \text{ has no bounded inverse on } X\}.$$

Without loss of generality we may assume that  $\alpha = 1$ . Our aim of this paper is to prove the following result

**Theorem IV.1.**

Let  $M$  be a Banach subspace of  $L^\infty(\mathbb{R}^+, X)$  that satisfies **(P1)**-**(P4)**. Suppose that  $SP(A, \mu)$  is at most countable. If  $u \in C_{ub}(\mathbb{R}^+, X)$  is a solution of equation (IV.1), then  $u \in M$ , provided that the function  $e^{-i\lambda t}u(t)$  is ergodic, for any  $\lambda \in SP(A, \mu)$ .

*Proof.* Let  $u$  be a solution of equation (IV.1) that satisfies the conditions of the theorem for any  $\lambda \in SP(A, \mu)$ . We can consider  $\mu$  as a measure on  $\mathbb{R}$  and supported on  $\mathbb{R}^+$ . There exists  $K, \omega > 0$  [8] such that

$$\|T(t)\| \leq Ke^{\omega t}, \quad t \in \mathbb{R}^+.$$

Let  $x_0 \in D(A)$ . We define the function  $w$  on  $\mathbb{R}^-$  by

$$w(t) = -e^{2\omega t}T(-t)x_0 - 2\omega e^{2\omega t} \int_0^{-t} T(s)\left(\frac{1}{2\omega}f(0) + x_0\right) ds + 2x_0, \quad t \leq 0.$$

We can see by [8, Theorem 2.4] that  $w$  is differentiable and  $w(t) \in D(A)$  for any  $t \leq 0$ . Set

$$h(t) = w'(t) - Aw(t), \quad t \leq 0.$$

We define the two functions  $v$  and  $g$  on  $\mathbb{R}$  by

$$v(t) = \begin{cases} u(t), & \text{if } t > 0 \\ w(t), & \text{if } t \leq 0. \end{cases}$$

$$g(t) = \begin{cases} f(t), & \text{if } t > 0 \\ h(t), & \text{if } t \leq 0. \end{cases}$$

Then  $v$  is a solution of the equation

$$v'(t) - Av(t) + \beta \int_0^t v(t-s) d\mu(s) = g(t), \quad t \in \mathbb{R}.$$

Since both of  $\lim_{t \rightarrow -\infty} v(t)$  and  $\lim_{t \rightarrow -\infty} g(t)$  exists, then both of  $v$  and  $g$  belongs to  $C_{ub}(\mathbb{R}, X)$ . The free term  $g \in \mathfrak{S}$ . Condition **(P4)**, stated above, implies that  $B\phi \in \mathfrak{S}$  for every  $B \in B(X)$  and every  $\phi \in \mathfrak{S}$ . Therefore,  $\sigma_{\mathfrak{S}}(v)$  is at most countable, by [7, Theorem II.1].

Now, we show that  $\sigma_M(u) \subseteq \sigma_{\mathfrak{S}}(v)$ . Indeed, let  $\gamma \in \sigma_M(u)$  and  $\psi \in L^1(\mathbb{R}, X)$  be such that  $\psi * v \in \mathfrak{S}$ . We have

$$\psi * v(t) = \psi * \tilde{u}(t) + \int_{\mathbb{R}^-} \psi(t-s)w(s) ds, \quad t \in \mathbb{R}.$$

Here  $\tilde{u}$  is the function in definition II.1. Since  $\lim_{t \rightarrow \infty} \int_{\mathbb{R} \leq 0} \psi(t-s)w(s) ds = 0$ , then the function  $\int_{\mathbb{R}^-} \psi(t-s)w(s) ds$  belongs to  $\mathfrak{S}$ , by lemma II.2. Hence,  $\psi * \tilde{u} \in \mathfrak{S}$  whence  $\psi \odot u \in M$  and  $\hat{\psi}(\gamma) = 0$ . So,  $\sigma_M(u) \subseteq \sigma_{\mathfrak{S}}(v)$  whence  $\sigma_M(u)$  is at most countable. We apply theorem III.2, to get  $u \in M$ .

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